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Regularity criterion to the axially symmetric Navier–Stokes equations

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ABSTRACT

Smooth solutions to the axially symmetric Navier–Stokes equations obey the following maximum principle: $\|ru_\theta(r,z,t)\|_{L^\infty} \leq \|ru_\theta(r,z,0)\|_{L^\infty}$. We first prove the global regularity of solutions if $\|ru_\theta(r,z,0)\|_{L^\infty}$ or $\|ru_\theta(r,z,t)\|_{L^\infty(r\leq r_0)}$ is small compared with certain dimensionless quantity of the initial data. This result improves the one in Zhen Lei and Qi S. Zhang [10]. As a corollary, we also prove the global regularity under the assumption that $|ru_\theta(r,z,t)| \leq |\ln r|^{-3/2}$, $\forall 0 < r \leq \delta_0 \in (0, 1/2)$.

1. Introduction

In the cylindrical coordinate system with $(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$, an axially symmetric solution of the Navier–Stokes equations is a solution of the following form

$$u(x,t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z,\ p(x,t) = p(r,z,t),$$

where

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right),\ e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right),\ e_z = (0,0,1).$$

In terms of (u_r, u_θ, u_z, p), the axially symmetric Navier–Stokes equations are as follows

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$$\begin{cases} \partial_t u_r + u\cdot\nabla u_r - \triangle u_r + \frac{u_r}{r^2} - \frac{u_\theta^2}{r} + \partial_r p = 0, \\ \partial_t u_\theta + u\cdot\nabla u_\theta - \triangle u_\theta + \frac{u_\theta}{r^2} + \frac{u_r u_\theta}{r} = 0, \\ \partial_t u_z + u\cdot\nabla u_z - \triangle u_z + \partial_z p = 0, \\ \partial_r(ru_r) + \partial_z(ru_z) = 0. \end{cases} \tag{1.1}$$

It is well-known that finite energy smooth solutions of the Navier–Stokes equations satisfy the following energy identity

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 \mathrm{d}s = \|u_0\|_{L^2}^2 < +\infty. \tag{1.2}$$

Denote $\Gamma = ru_\theta$. One can easily check that

$$\partial_t\Gamma + u\cdot\nabla\Gamma - \triangle\Gamma + \frac{2}{r}\partial_r\Gamma = 0. \tag{1.3}$$

A significant consequence of (1.3) is that smooth solutions of the axially symmetric Navier–Stokes equations satisfy the following maximum principle (see, for instance, [1,3])

$$\|\Gamma\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}. \tag{1.4}$$

We can compute the vorticity

$$\omega = \nabla\times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,$$

where

$$\omega_r = -\partial_z(u_\theta),\ \omega_\theta = \partial_z(u_r) - \partial_r(u_z),\ \omega_z = \frac{1}{r}\partial_r(ru_\theta).$$

Denote

$$\Omega = \frac{\omega_\theta}{r},\ J = \frac{\omega_r}{r} = -\frac{\partial_z u_\theta}{r},$$

then

$$\begin{cases} \partial_t\Omega + u\cdot\nabla\Omega - \left(\triangle + \frac{2}{r}\partial_r\right)\Omega + 2\frac{u_\theta}{r}J = 0, \\ \partial_t J + u\cdot\nabla J - \left(\triangle + \frac{2}{r}\partial_r\right)J - (\omega_r\partial_r + \omega_z\partial_z)\frac{u_r}{r} = 0. \end{cases} \tag{1.5}$$

We emphasize that J was introduced by Chen–Fang–Zhang in [2], while Ω appeared much earlier and can be at least tracked back to the book of Majda–Bertozzi in [12]. Both of the two new variables are of great importance in our work.

Our goal is to prove that the smallness of $\|\Gamma\|_{L^\infty(r\le r_0)}$ or $\|\Gamma_0\|_{L^\infty}$ implies the global regularity of the solutions. Here is our result.

Theorem 1.1. *Let $r_0 > 0$. Suppose that $u_0 \in H^2$ such that $\Gamma_0 \in L^\infty$. Denote*

$$M_1 = (1 + \|\Gamma_0\|_{L^\infty})\|u_0\|_{L^2} \text{ and } M_0 = (\|J_0\|_{L^2} + \|\Omega_0\|_{L^2})M_1^3.$$