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Nonlocal Schrödinger equations in metric measure spaces ☆

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A B S T R A C T

In this note we consider the pointwise convergence to the initial data for the solutions of some nonlocal dyadic Schrödinger equations on spaces of homogeneous type. We prove the a.e. convergence when the initial data belongs to a dyadic version of an L^2 based Besov space.

1. Introduction

In quantum mechanics the position of a particle in the *space* is described by the probability density function $|\varphi|^2 = \varphi\overline{\varphi}$ where φ is a solution of the *Schrödinger equation*. In the classical free particle model, the *space* is the Euclidean and the *Schrödinger equation* is associated to the Laplace operator, i.e. $i\frac{\partial \varphi}{\partial t} = \triangle\varphi$. Hence the probability of finding the particle inside the Borel set E of the Euclidean space at time t is given by $\int_E |\varphi(x,t)|^2\, dx$.

The pointwise convergence to the initial data for the classical Schrödinger equation in Euclidean settings is a hard problem. It is well known that some regularity in the initial data is needed [6,9,11,8,19,22,20].

Nonlocal operators instead of the Laplacian in this basic model have been considered previously in the Euclidean space (see for example [15] and references in [21]). The nonlocal fractional derivatives as substitutes of the Laplacian become natural objects when the space itself lacks any differentiable structure and only an analysis of order less than one can be carried out.

We shall be brief in our introduction of the basic setting. For a more detailed approach see [1]. Let (X, d, μ) be a space of homogeneous type (see [17]). Let $\mathscr{D}$ be a dyadic family in X as constructed by

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M. Christ in [7]. Let $\mathscr{H}$ be a Haar system for $L^2(X,\mu)$ associated to $\mathscr{D}$ as built in [2] (see also [5,3]). Following the basic notation introduced in Section 1, we shall use $\mathscr{H}$ itself as the index set for the analysis and synthesis of signals defined on X. Precisely, by $Q(h)$ we denote the member of $\mathscr{D}$ on which h *is based*. With $j(h)$ we denote the integer scale j for which $Q(h) \in \mathscr{D}^j$.

The system $\mathscr{H}$ is an orthonormal basis for L^2_0, where L^2_0 coincides with $L^2(X,\mu)$ if $\mu(X) = +\infty$ and $L^2_0 = \{f \in L^2 : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$. For a given $Q \in \mathscr{D}$ the number of wavelets h based on Q is $\#\vartheta(Q) - 1$, where $\vartheta(Q)$ is the offspring of Q and $\#\vartheta(Q)$ is its cardinal. The homogeneity property of the space together with the metric control of the dyadic sets guarantees a uniform upper bound for $\#\vartheta(Q)$. On the other hand $\#\vartheta(Q) \geq 1$ for every $Q \in \mathscr{D}$.

Let $(X, d, \mu, \mathscr{D}, \mathscr{H})$ be given as before. For the sake of simplicity we shall assume along this paper that X itself is a quadrant for $\mathscr{D}$. We say that X itself is a quadrant if any two cubes in X have a common ancestor. A distance in X associated to $\mathscr{D}$ can be defined by $\delta(x,y) = \min\{\mu(Q) : Q \in \mathscr{D}$ such that $x, y \in Q\}$ when $x \neq y$ and $\delta(x,x) = 0$. The next lemma, borrowed from [1], reflects the one dimensional character of X equipped with δ and μ.

Lemma 1. *(See Lemma 3.1 in [1].) Let $0 < \varepsilon < 1$, and let Q be a given dyadic cube in X. Then, for $x \in Q$, we have*

$$\int_{X \setminus Q} \frac{d\mu(y)}{\delta(x,y)^{1+\varepsilon}} \simeq \mu(Q)^{-\varepsilon}.$$

Furthermore the integral of $\delta^{-1}(x,\cdot)$ diverges on each dyadic cube containing x and, when the measure of X is not finite, on the complement of each dyadic cube.

For a complex value function f Lipschitz continuous with respect to δ define

$$D^\beta f(x) = \int_X \frac{f(x) - f(y)}{\delta(x,y)^{1+\beta}} d\mu(y).$$

One of the key results relating the operator D^β with the Haar system is provided by the following spectral theorem contained in [1].

Theorem 2. *(See Theorem 3.1 in [1].) Let $0 < \beta < 1$. For each $h \in \mathscr{H}$ we have*

$$D^\beta h(x) = m_h \mu(Q(h))^{-\beta} h(x), \tag{1}$$

where m_h is a constant that may depend on $Q(h)$ but there exist two finite and positive constants M_1 and M_2 such that

$$M_1 < m_h < M_2, \qquad \text{for all } h \in \mathscr{H}. \tag{2}$$

Set $B^\lambda_2(X, \delta, \mu)$ to denote the space of those functions $f \in L^2(X,\mu)$ satisfying

$$\iint_{X \times X} \frac{|f(x) - f(y)|^2}{\delta^{1+2\lambda}(x,y)} d\mu(x) d\mu(y) < \infty.$$

The projection operator defined on L^2 onto V_0 the subspace of functions which are constant on each cube $Q \in \mathscr{D}^0$ is denoted by P_0. We are now in position to state the main results of this paper.