J. Math. Anal. Appl. 435 (2016) 440–460

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Approach for a metric space with a convex combination operation and applications

Nguyen Tran Thuan

Department of Mathematics, Vinh University, Nghe An Province, Viet Nam

A R T I C L E I N F O

Article history:
Received 18 March 2015
Available online 3 October 2015
Submitted by U. Stadtmueller

Keywords:
Convex combination
Embedding
Ergodic theorem
Jensen's inequality
Metric space

A B S T R A C T

In this study, we embed a metric space endowed with a convex combination operation, which is called a convex combination space, into a Banach space, where the embedding preserves the structures of the metric and convex combination. We also establish applications of this embedding for a random element that takes values in this type of space. On the one hand, we show some useful properties of mathematical expectation, such as the representation of expectation through continuous affine mappings and the linearity of expectation. On the other hand, the notion of conditional expectation is also introduced and discussed. Using this embedding theorem, we establish some basic properties of conditional expectation, Jensen's inequality, the convergences of martingales, and ergodic theorem.

1. Introduction

Probability theory in linear spaces has been considered for many years and it has been extended to more general models that are nonlinear, such as hyperspaces of linear space or metric spaces in general. Basic objects such as expectation or the conditional expectation of random elements that take values in a metric space have also attracted the attention of many researchers. The first researcher to introduce a concept of mathematical expectation for a random element with values in a metric space was probably Doss [4] in 1949. Subsequently, many different definitions of expectation and conditional expectation have been provided in different types of metric spaces using various methods, including the studies by Émery and Mokobodzki [5], Herer [8–10], Raynaud de Fitte [17], Sturm [3,18], and the monograph by Molchanov [11].

In 2006, Terán and Molchanov [21] introduced the concept of a convex combination (CC) space and the class of these spaces is larger than the class of Banach spaces, the class of hyperspace of compact subsets, and the class of upper semicontinuous functions (also called fuzzy sets) with compact support in Banach space [21]. In addition, they provided many interesting illustrative examples of this concept, e.g., the space of all cumulative distribution functions or the space of upper semicontinuous functions with t-norm. A CC

E-mail address: thuan.nguyen@vinhuni.edu.vn.

http://dx.doi.org/10.1016/j.jmaa.2015.09.083
0022-247X/

space is a metric space endowed with a CC operation and the extension from linear space to CC space is not trivial. Some very basic sets, such as singletons and balls, may not be convex in CC space, which might not match with our usual intuition, but this occurs in many practical models. For example, if we consider the hyperspace of all compact subsets of Banach space where the CC operation is generated by Minkovski addition and scalar multiplication, then $\lambda A + (1-\lambda)A$ is not equal to A unless A is convex, which means that A is a non-convex singleton in this space. Another example is the space of integrable probability distributions, where the CC operation is generated by the convolution operation (see [20,21]). The expected value for a random element that takes values in CC space was constructed by Terán and Molchanov. This notion of expectation extended the corresponding concept that considers Banach space as well as the hyperspace of compact subsets. Furthermore, the authors established the Etemadi strong law of large numbers (SLLN) for normalized sums of pairwise independent, identically distributed (i.i.d.) random elements in this type of space ([21], Theorem 5.1), and other applications can be found in [15,20,22].

A CC space may have many singletons that are not convex, but it always contains a subspace (we will call convexifiable domain) where every singleton and ball is convex, while it was shown in [21] that this subspace has some properties resembling linearity. Therefore, it is natural to ask whether this convexifiable domain can be embedded isometrically into some normed linear space such that the convex combination structure is preserved. It should be noted that the expectation of every integrable random element that takes values in a CC space always belongs to this convexifiable domain. Therefore, if the embedding is established, we will have more tools to explore this type of expectation as well as the properties of CC space.

In this study, we address the question stated above. In particular, we show that the convexifiable domain of a complete CC space can be embedded into a Banach space such that the embedding is isometric and the structure of convex combination is preserved (as demonstrated in Section 3).

The main applications of the approach via embedding theorem are presented in Section 4. On the one hand, we give some nice properties of mathematical expectation, including the representation of the expectation through continuous affine mappings and Jensen's inequality (which was first proved by Terán [20], but we prove this again in the present study using another method). On the other hand, we introduce and discuss the notion of conditional expectation for integrable random elements that take values in CC space. Using embedding theorem, we establish some basic properties of conditional expectation, Jensen's inequality, the convergences of martingales, and ergodic theorem.

Finally, some miscellaneous applications and remarks are given in Section 5.

2. Preliminaries

For the reader's convenience, we now present a short introduction to the approach given by Terán and Molchanov in [21]. Let $(\mathfrak{X}, d)$ be a metric space and for $u, x \in \mathfrak{X}$, we denote $\|x\|_u := d(u, x)$. Based on $\mathfrak{X}$, introduce a *convex combination operation*, which for all $n \geqslant 2$, numbers $\lambda_1, \ldots, \lambda_n > 0$ that satisfy $\sum_{i=1}^n \lambda_i = 1$ and all $u_1, \ldots, u_n \in \mathfrak{X}$, this operation produces an element of $\mathfrak{X}$, which is denoted by $[\lambda_1, u_1; \ldots; \lambda_n, u_n]$ or $[\lambda_i, u_i]_{i=1}^n$. Note that $[\lambda_1, u_1; \ldots; \lambda_n, u_n]$ and the shorthand $[\lambda_i, u_i]_{i=1}^n$ have the same intuitive meaning as the more familiar $\lambda_1 u_1 + \cdots + \lambda_n u_n$ and $\sum_{i=1}^n \lambda_i u_i$, but $\mathfrak{X}$ is not assumed to have any addition or multiplication. Suppose that $[1, u] = u$ for every $u \in \mathfrak{X}$ and that the following properties are satisfied:

(CC.i) (Commutativity) $[\lambda_i, u_i]_{i=1}^n = [\lambda_{\sigma(i)}, u_{\sigma(i)}]_{i=1}^n$ for every permutation σ of $\{1, \ldots, n\}$;
(CC.ii) (Associativity) $[\lambda_i, u_i]_{i=1}^{n+2} = \left[\lambda_1, u_1; \ldots; \lambda_n, u_n; \lambda_{n+1} + \lambda_{n+2}, \left[\frac{\lambda_{n+j}}{\lambda_{n+1}+\lambda_{n+2}}, u_{n+j}\right]_{j=1}^2\right]$;
(CC.iii) (Continuity) If $u, v \in \mathfrak{X}$ and $\lambda^{(k)} \to \lambda \in (0;1)$ as $k \to \infty$, then $[\lambda^{(k)}, u; 1-\lambda^{(k)}, v] \to [\lambda, u; 1-\lambda, v]$;
(CC.iv) (Negative curvature) If $u_1, u_2, v_1, v_2 \in \mathfrak{X}$ and $\lambda \in (0, 1)$, then

$$d([\lambda, u_1; 1-\lambda, u_2], [\lambda, v_1; 1-\lambda, v_2]) \leqslant \lambda d(u_1, v_1) + (1-\lambda) d(u_2, v_2);$$