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A coincidence point theorem for sequentially continuous mappings

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ABSTRACT

The aim of this paper is to present a coincidence point theorem for sequentially weakly continuous maps. Moreover, as a consequence, a critical point theorem for functionals possibly containing a nonsmooth part is obtained. Finally, as an application, existence results for nonlinear differential problems depending also on the derivative of the solution are established.

1. Introduction

In the first half of the eighties, Arino, Gautier and Penot [1] have made the useful observation that the sequential weak continuity is sufficient in order to apply the Schauder–Tychonoff fixed point theorem, without need to require the stronger assumption of weak continuity. A decade later, inspired by [1], Bonanno and Marano [2] have used a similar remark regarding the fixed point theorem of Fan [4] obtaining a fixed point theorem for multifunctions where the key assumption is that the graph is sequentially weakly closed (see Theorem 2.1 below).

The aim of this short note is to present a coincidence point theorem for sequentially weakly continuous functions, by exploiting ideas contained in the above mentioned results. Our main result is Theorem 2.2, where the existence of a coincidence point for mappings between two Banach spaces, possibly multivalued, is established. Moreover, as a consequence, the existence of a critical point in a possibly nonsmooth setting is obtained (see Theorem 2.3). As an application, the existence of solutions for a two point nonlinear boundary

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value problem is investigated (see Theorem 3.1). It is worth noticing that in our result the nonlinear term may also depend on the derivative, unlike some existence results obtained through the variational methods (see Remark 3.2).

As an example, we present here a special case of Theorem 3.1.

Theorem 1.1. *Let $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]-\lambda^*, \lambda^*[$, problem*

$$\begin{cases} -u'' = \lambda g(t, u, u') & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least one classical solution.

Further applications of our abstract result concern semilinear elliptic problems with mixed boundary conditions (see Theorems 4.1 and 4.2).

The paper is organized as follows. In Section 2, we present a coincidence point theorem and some consequences. In Section 3, we establish an existence result for a two point boundary value problem, while in Section 4 we focus on semilinear elliptic problems with boundary conditions.

2. Fixed point and coincidence theorems

The following result needed in the sequel is a special case of [2, Theorem 2.1], which is obtained from the classical fixed point theorem of Ky Fan [4, Theorem 1].

Theorem 2.1. *Let X be a real Banach space and let K be a weakly compact, convex subset of X. Suppose that Φ is a multifunction from K into itself with nonempty convex values and sequentially weakly closed graph. Then there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$.*

Remark 2.1. If Φ is a function, Theorem 2.1 is a fixed point theorem for maps. In this case, if Φ is sequentially weakly continuous, then it has a sequentially weakly closed graph, so Theorem 2.1 retrieves the fixed point theorem of Arino, Gautier and Penot [1, Theorem 1], which in turn is obtained from the classical Schauder–Tychonoff's theorem.

Now we present the main result of this section, which is a coincidence point theorem.

Theorem 2.2. *Let X, Y be real Banach spaces, let K be a weakly compact, convex subset of X, and let $F, G : K \to 2^Y$ be multifunctions such that*

(*i*) *for every $x \in K$, the set*

$$F^-(G(x)) := \{u \in K : G(x) \cap F(u) \neq \emptyset\}$$

is nonempty and convex;

(*ii*) *the graphs of F and G are sequentially weakly closed, and $F(K)$ is sequentially weakly compact.*

Then there exists $x_0 \in K$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.

Proof. Using assumption (i), the multifunction $\Phi : K \to 2^K$ defined by