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Modulus of continuity with respect to semigroups of analytic functions and applications

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ABSTRACT

Given a complex Banach space E, a semigroup of analytic functions (φ_t) and an analytic function $F : \mathbb{D} \to E$ we introduce the modulus $w_\varphi(F,t) = \sup_{|z|<1} \|F(\varphi_t(z)) - F(z)\|$. We show that if $0 < \alpha \leq 1$ and F belongs to the vector-valued disc algebra $A(\mathbb{D}, E)$, the Lipschitz condition $M_\infty(F', r) = O((1-r)^{1-\alpha})$ as $r \to 1$ is equivalent to $w_\varphi(F,t) = O(t^\alpha)$ as $t \to 0$ for any semigroup of analytic functions (φ_t), with $\varphi_t(0) = 0$ and infinitesimal generator $\mathcal{G}$, satisfying that φ_t' and $\mathcal{G}$ belong to $H^\infty(\mathbb{D})$ with $\sup_{0\leq t\leq 1} \|\varphi'\|_\infty < \infty$, and in particular is equivalent to the condition $\|F - F_r\|_{A(\mathbb{D},E)} = O((1-r)^\alpha)$ as $r \to 1$. We apply this result to particular semigroups (φ_t) and particular spaces of analytic functions E, such as Hardy or Bergman spaces, to recover several known results about Lipschitz type functions.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the Fréchet space of all analytic functions in the unit disk endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. A Banach space $(X, \|\cdot\|)$ such that $X \subset \mathcal{H}(\mathbb{D})$ with continuous inclusion will be called a *Banach space of analytic functions*. We shall say that X is a *homogeneous Banach space of analytic functions* (see [3,13]) if it satisfies the following properties

$$A(\mathbb{D}) \subset X \subset \mathcal{H}(\mathbb{D}) \tag{1}$$

with continuous inclusions,

$$f \in X \Longrightarrow f_\xi \in X \text{ and } \|f_\xi\| = \|f\| \text{ for any } |\xi| = 1, \tag{2}$$

and there exists $C > 0$ such that

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$$f \in X \Longrightarrow f_r \in X \text{ and } \|f_r\| \leq C\|f\| \text{ for any } 0 \leq r < 1, \tag{3}$$

where $f_z(w) = f(zw)$ for $z \in \overline{\mathbb{D}}$ and $A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap H^\infty$ stands for the disc algebra, that is the closed subspace of bounded analytic functions with continuous extensions to the boundary.

Particular examples are, for $1 \leq p \leq \infty$, the Hardy spaces H^p with the norm

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r)$$

where $M_p(f,r) = (\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi})^{1/p}$ (or $M_\infty(f,r) = \sup_{|\xi|=1} |f(r\xi)|$) and the Bergman spaces A^p with the norm

$$\|f\|_{A^p} = (\int_{\mathbb{D}} |f(w)|^p dA(w))^{1/p}$$

where $dA(w)$ stands for the normalized Lebesgue measure in $\mathbb{D}$.

If $(X, \|\cdot\|)$ is a homogeneous Banach space of analytic functions and $f \in X$ we shall denote

$$w_X(f,t) = \sup_{|\theta|\leq t} \|f_{e^{i\theta}} - f\| \text{ and } M_X(f,r) = \sup_{0\leq\delta\leq r} \|f_\delta\|.$$

We keep the classical notations $w_p(f,t) = w_{H^p}(f,t)$ and $M_p(f,r) = M_{H^p}(f,r)$ for $f \in H^p$. It is easy to see (due to the density of analytic polynomials in the spaces $A(\mathbb{D})$, H^p and A^p) that $w_p(f,t) \to 0$ and $w_{A^p}(f,t) \to 0$ as $t \to 0$ for any f in the corresponding space. It is also well known (see for instance [6,9,16]) that $\|f - f_r\|_{H^p} \to 0$ and $\|f - f_r\|_{A^p} \to 0$ as $r \to 1$ for $1 \leq p < \infty$. It goes back to the work of Hardy and Littlewood and further extensions (see for instance [6,12,15]) that for each $1 \leq p \leq \infty$, $0 < \alpha \leq 1$ and $f \in H^p$ the following conditions are equivalent:

(a) $w_p(f,t) = O(t^\alpha)$, $t \to 0$.
(b) $\|f - f_r\|_{H^p} = O((1-r)^\alpha)$, $r \to 1$.
(c) $M_p(f', r) = O((1-r)^{\alpha-1})$, $r \to 1$.

Of course this type of result can also be rephrased in terms of fractional derivatives. The reader is referred to the work of H. Komatsu [10, Theorem 13.8] for an alternative proof of such a result which uses the heavy machinery of fractional powers of operators (see also [11] for more general results in this direction).

In a recent paper Galanopoulos, Siskakis and Stylogiannis (see [8, Theorem 4.1]) have shown the following analogue for Bergman spaces, namely for $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in A^p$ the following are equivalent

(i) $w_{A^p}(f,t) = O(t^\alpha)$, $t \to 0$
(ii) $\|f - f_r\|_{A^p} = O((1-r)^\alpha)$, $r \to 1$
(iii) $A_p(f', r) = O((1-r)^{\alpha-1})$, $r \to 1$ where $A_p(f,r) = \|f_r\|_{A^p}$.

There are three goals in the paper: First to exhibit that the equivalences between (a), (b) and (c) and between (i), (ii) and (iii) for Hardy spaces H^p and Bergman spaces A^p in the case $1 \leq p < \infty$ actually follow from the case $p = \infty$ with the use of vector-valued functions in the disc algebra, second to show that they hold true not only for Hardy and Bergman spaces but also for any homogeneous Banach space of analytic functions and third to add other equivalent formulations in terms of

$$w_\varphi(F,t) = \sup_{|z|<1} \|F(\varphi_t(z)) - F(z)\|$$