J. Math. Anal. Appl. 435 (2016) 646–660

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stabilized low order finite elements for Stokes equations with damping ☆

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ARTICLE INFO

Article history:
Received 16 January 2015
Available online 26 October 2015
Submitted by W.L. Wendland

Keywords:
Stokes equations
Damped term
Stabilization methods
Low order elements
Error estimates

ABSTRACT

In this paper, we consider the stationary Stokes equations with damping. Low order mixed finite element spaces are used to approximate the velocity and the pressure, and a local pressure projection stabilization method is used for the pairs to overcome the lack of the *inf-sup* condition. The stability of this method is proved, and the optimal order error estimates are derived by some nonlinear analysis techniques. At last, two numerical examples are implemented to test the stability and effectiveness of the proposed method.

1. Introduction

In this paper, we consider the following stationary Stokes equations with damping in $\mathbb{R}^n$ $(n = 2, 3)$: find (u, p) such that

$$\begin{cases} -\nu\Delta u + \alpha|u|^{r-2}u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $u = (u_1, u_2, \cdots, u_n)$ and p are the fluid velocity and pressure, respectively, ν is the viscosity coefficient of the flow, $1 < r < \infty$ and $\alpha > 0$ are two damping parameters, respectively, and f is a given external force. Stokes equations with damping are widely used in geophysics and ocean acoustics [1,17]. The damping comes from the resistance to the motion of the flow, which describes various physical situations such as porous media flow, drag, or friction effects and some dissipative mechanisms [7,8].

☆ This work was supported by National Basic Research Program of China (973 Program: 2011CB013800) and National Natural Science Fund of China (no. 10971203; no. 11271340).

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http://dx.doi.org/10.1016/j.jmaa.2015.10.040

Some studies have already been devoted to the theoretical and numerical analysis of the problems with damping. In [10], the existence of the strong and weak solutions of the Navier–Stokes equations with nonlinear damping $\alpha|u|^{r-1}u(r > 0)$ and the large time behavior of weak solutions were studied. In [27], the superclose and superconvergence phenomenon of some stable mixed finite elements were studied. In fact, there are many stable mixed finite element spaces for the Stokes problem, such as Hood–Taylor element [28], the Mini element [2], Bernardi–Raugel element [4] and so on. These elements are also stable for the Stokes problem with damping.

On the other hand, the equal order finite element spaces are very popular in the engineering practice, since these elements have a very convenient numerical implementation. However, these element pairs usually do not satisfy the *inf-sup* condition. In order to overcome this drawback, these pairs are usually supplemented by the stabilization methods. Consistently stabilization methods accomplish this by adding the residuals of the momentum equation to the original discrete schemes, such as the SUPG stabilization [16,20], the Galerkin least-squares stabilization [19] and the Douglas–Wang stabilization [14]. These stabilization methods usually depend on the stabilization parameters. In addition, for low-order pairs, the pressure and velocity derivatives in the residual term either vanish or are poorly approximated. Of course, there are several stabilization methods without using residuals. A very natural choice is the Brezzi–Pitkäranta stabilization method [9], but this method usually leads to a spurious homogeneous Neumann type boundary condition for the pressure. Some other methods are the pressure gradient projection (PGP) stabilization [12,13] and the local pressure gradient projection stabilization (LPS) [3,25]. In the PGP methods, the pressure gradient is projected onto the continuous velocity space, and this gives rise to a globally coupled problem, while in the LPS methods, the pressure gradient is projected onto an element patch space, which leads to a local problem.

Recently, a local pressure projection stabilization was developed for low order mixed finite elements in [6,18]. This stabilized method is parameter free and unconditionally stable, does not require the calculation of higher order derivatives or edge-based data structures, and always leads to symmetric linear systems. This stabilization method has been applied to many problems, such as Navier–Stokes equations [15,21,23], Darcy equations [5], the primitive equations of the ocean [11], contact problem [24] and so on, but for the Stokes equations with damping, there are no researches about this method. In this paper, we will try to apply the stabilization method to this problem. Compared with Navier–Stokes equations, Stokes equations with damping are more complicated for the nonlinear damping term $\alpha|u|^{r-2}u$, while Navier–Stokes equations have a linear damping term. Some nonlinear analysis techniques are used in the error estimates, and the Picard iteration method is employed for the nonlinear damping term in the numerical implementation.

The rest of this paper is organized as follows. In section 2, we introduce some notations used throughout the paper, the mixed variational formulation of this problem, and a weaker form of the *inf-sup* condition that holds for the spaces of interest to us. In section 3, we give the stabilization scheme and then prove the stability. In section 4, we deal with the error analysis. In section 5, we implement two numerical examples to test the stability and accuracy of the stabilization method. In section 6, we give a conclusion.

2. Mixed finite element methods

Firstly, we introduce some notations and function spaces. Ω is a bounded and convex domain in $\mathbb{R}^n$. For integers $l > 0$ and $m \geq 0$, $L^l(\Omega)$ is the space of functions for which the l-th power of the absolute value is integrable on Ω, and $H^m(\Omega)$ is the spaces of functions whose derivatives up to the order m belong to $L^2(\Omega)$. $\|\cdot\|_{0,l}$ and $\|\cdot\|_m$ denote the Sobolev norm on $L^l(\Omega)$ and $H^m(\Omega)$, respectively. $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m$. h is the mesh size. C is a positive constant whose value may change from place to place but remains independent of h. c with subscript is a positive constant independent of h.