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Uniform polynomial approximation with A^* weights having finitely many zeros

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ABSTRACT

We prove matching direct and inverse theorems for uniform polynomial approximation with A^* weights (a subclass of doubling weights suitable for approximation in the $\mathbb{L}_\infty$ norm) having finitely many zeros and not too "rapidly changing" away from these zeros. This class of weights is rather wide and, in particular, includes the classical Jacobi weights, generalized Jacobi weights and generalized Ditzian–Totik weights. Main part and complete weighted moduli of smoothness are introduced, their properties are investigated, and equivalence type results involving related realization functionals are discussed.

1. Introduction

Recall that a nonnegative integrable function w is a doubling weight (on $[-1, 1]$) if there exists a positive constant L (a so-called doubling constant of w) such that

$$w(2I) \le Lw(I), \tag{1.1}$$

for any interval $I \subset [-1, 1]$. Here, $2I$ denotes the interval of length $2|I|$ ($|I|$ is the length of I) with the same center as I, and $w(I) := \int_I w(u)du$. Note that it is convenient to assume that w is identically zero outside $[-1, 1]$ which allows us to write $w(I)$ for any interval I that is not necessarily contained in $[-1, 1]$. Let $\mathcal{D}W_L$ denote the set of all doubling weights on $[-1, 1]$ with the doubling constant L, and $\mathcal{D}W := \cup_{L>0}\mathcal{D}W_L$, *i.e.*, $\mathcal{D}W$ is the set of all doubling weights.

It is easy to see that $w \in \mathcal{D}W_L$ if and only if there exists a constant $\kappa \ge 1$ such that, for any two adjacent intervals $I_1, I_2 \subset [-1, 1]$ of equal length,

$$w(I_1) \le \kappa w(I_2). \tag{1.2}$$

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Clearly, κ and L depend on each other. In fact, if $w \in \mathcal{D}W_L$ then (1.2) holds with $\kappa = L^2$. Conversely, if (1.2) holds, then $w \in \mathcal{D}W_{1+\kappa}$.

Following [6,7], we say that w is an A^* weight (on $[-1,1]$) if there is a constant L^* (a so-called A^* constant of w) such that, for all intervals $I \subset [-1,1]$ and $x \in I$, we have

$$w(x) \le \frac{L^*}{|I|} w(I). \tag{1.3}$$

Throughout this paper, $A^*_{L^*}$ denotes the set of all A^* weights on $[-1,1]$ with the A^* constant L^*. We also let $A^* := \cup_{L>0} A^*_{L^*}$, *i.e.*, A^* is the set of all A^* weights. Note that any A^* weight is doubling, *i.e.*, $A^*_{L^*} \subset \mathcal{D}W_L$, where L depends only on L^*. This was proved in [7] and is an immediate consequence of the fact (see [7, Theorem 6.1]) that if $w \in A^*_{L^*}$ then, for some l depending only on L^* (for example, $l = 2L^*$ will do), $w(I_1) \ge (|I_1|/|I_2|)^l w(I_2)$, for all intervals $I_1, I_2 \subset [-1,1]$ such that $I_1 \subset I_2$. Indeed, for any $I \subset [-1,1]$, this implies $w(I) \ge (|I|/|2I \cap [-1,1]|)^l\, w(2I) \ge 2^{-l} w(2I)$, which shows that $w \in \mathcal{D}W_{2^l}$.

Moreover, it is known and is not difficult to check (see [7, pp. 58 and 68]) that all A^* weights are A_∞ weights. Here, A_∞ is the union of all Muckenhoupt A_p weights and can be defined as the set of all weights w such that, for any $0 < \alpha < 1$, there is $0 < \beta < 1$ so that $w(E) \ge \beta w(I)$, for all intervals $I \subset [-1,1]$ and all measurable subsets $E \subset I$ with $|E| \ge \alpha |I|$ (see *e.g.* [10, Chapter V]).

Clearly, any A^* weight on $[-1,1]$ is bounded since if $w \in A^*_{L^*}$, then $w(x) \le L^* w[-1,1]/2$, $x \in [-1,1]$. (We slightly abuse the notation and write $w[a,b]$ instead of $w([a,b])$ throughout this paper.) At the same time, not every bounded doubling weight is an A^* weight (for example, the doubling weight constructed in [2] is bounded and is not in A_∞, and so it is not an A^* weight either).

Throughout this paper, we use the standard notation $\|f\|_I := \|f\|_{\mathbb{L}_\infty(I)} := \operatorname{ess\,sup}_{u\in I} |f(u)|$ and $\|f\| := \|f\|_{[-1,1]}$. Also,

$$E_n(f, I)_w := \inf_{q \in \Pi_n} \|w(f-q)\|_I ,$$

where Π_n is the space of algebraic polynomials of degree $\le n-1$.

The following theorem is due to G. Mastroianni and V. Totik [8, Theorem 1.4] and is the main motivation for the present paper (see also [5–7]).

Theorem A. *(See [8, Theorem 1.4].) Let $r \in \mathbb{N}$, $M \ge 3$, $-1 = z_1 < \cdots < z_M = 1$, and let w be a bounded generalized Jacobi weight*

$$w_{\mathcal{J}}(x) := \prod_{j=1}^{M} |x - z_j|^{\gamma_j} \quad \text{with } \gamma_j \ge 0,\ 1 \le j \le M. \tag{1.4}$$

Then there is a constant c depending only on r and the weight w such that, for any f,

$$E_n(f, [-1,1])_{w_{\mathcal{J}}} \le c\,\omega^r_\varphi(f, 1/n)^*_{w_{\mathcal{J}}},$$

and

$$\omega^r_\varphi(f, 1/n)^*_{w_{\mathcal{J}}} \le c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f, [-1,1])_{w_{\mathcal{J}}},$$

where

$$\omega^r_\varphi(f,t)^*_{w_{\mathcal{J}}} := \sum_{j=1}^{M-1} \sup_{0<h\le t} \left\| w_{\mathcal{J}}(\cdot) \Delta^r_{h\varphi(\cdot)}(f, \cdot, J_{j,h}) \right\| + \sum_{j=1}^{M} E_r(f, I_{j,t})_{w_{\mathcal{J}}}$$