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Subdivisions of rotationally symmetric planar convex bodies minimizing the maximum relative diameter

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ABSTRACT

In this work we study subdivisions of k-rotationally symmetric planar convex bodies that minimize the maximum relative diameter functional. For some particular subdivisions called k-partitions, consisting of k curves meeting in an interior vertex, we prove that the so-called *standard k-partition* (given by k equiangular inradius segments) is minimizing for any $k \in \mathbb{N}$, $k \geqslant 3$. For general subdivisions, we show that the previous result only holds for $k \leqslant 6$. We also study the optimal set for this problem, obtaining that for each $k \in \mathbb{N}$, $k \geqslant 3$, it consists of the intersection of the unit circle with the corresponding regular k-gon of certain area. Finally, we also discuss the problem for planar convex sets and large values of k, and conjecture the optimal k-subdivision in this case.

1. Introduction

The study of the classical geometric magnitudes associated to a planar compact convex set (perimeter, area, inradius, circumradius, diameter and minimal width) is one of the main points of interest for Convex Geometry, and these magnitudes have been considered since ancient times. They appear in the origin of this mathematical field, and in fact, there is a large variety of related problems, providing a more complete understanding of these geometric measures.

One of the most common problems involving these magnitudes consists of finding precise relations between some of them (and not only in the convex framework). For instance, the well-known isoperimetric inequality relates the perimeter and the area functionals, stating that, among all planar sets with fixed enclosed area, the circle is the one with the least possible perimeter [16,14]. In the same direction, the isodiametric inequality asserts that the circle is also the planar compact convex set of prescribed area with the minimum possible diameter [1]. Another example is given by the equilateral triangle, which is the planar compact

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convex set with minimum enclosed area for fixed minimal width [15]. We point out that these relations are usually established by means of appropriate general inequalities, with the characterization of the equality case providing then the optimal set for the corresponding functionals (that is, the set attaining the minimum possible value for the functional). A nice detailed description of the relations between two (and three) of these classical magnitudes can be found in [17].

Apart from the above approach, these geometric functionals can be studied in a large range of different related problems. One of these problems, which has been considered in several works during the last years, is the so-called *fencing problem*, or more generally, the *partitioning problem*. For these problems, the main interest is determining the division of the given set into k connected subsets (of equal or unequal areas) which minimizes (or maximizes) a particular geometric magnitude [7]. The relative isoperimetric problem is one of these questions, where the goal is minimizing the perimeter of the division curves for fixed enclosed areas. For this problem, the general geometric properties satisfied by the minimizing divisions can be deduced by using a variational approach, and particular results characterizing the solutions have been obtained for some regular polygons and for the circle, see [19,2,4,5].

Regarding the diameter functional, the partitioning problem can be studied by means of the *maximum relative diameter* functional. We recall that for a planar compact convex set C and a decomposition of C into k connected subsets $C_1, \dots, C_k$, the maximum relative diameter associated to the decomposition is defined as the maximum of the diameters of the subsets C_i, for $i = 1, \dots, k$ (this means that it measures the largest distance in the subsets provided by the decomposition). In this setting, the partitioning problem seeks the decomposition of C into k subsets with the least possible value for the maximum relative diameter. This problem has been recently studied for $k = 2$ and $k = 3$ under some additional symmetry hypotheses [11,3], and the results therein constitute the main motivation for our work.

In [11], the previous problem is treated for *centrally symmetric* planar compact convex sets, and for decompositions into two subsets of equal areas, proving that the minimizing division for the maximum relative diameter is always given by a line segment passing through the center of symmetry of the set [11, Prop. 4]. We point out that a more precise characterization of such a minimizing segment is not provided. Furthermore, the optimal set for this problem (that is, the centrally symmetric planar compact convex set attaining the minimum possible value for the maximum relative diameter) is also determined up to dilations, consisting of a certain intersection of two unit circles and a strip bounded by two parallel lines, see [11, Th. 5].

In [3], the problem is considered for 3-*rotationally symmetric* planar compact convex sets, and for divisions into three subsets of equal areas. The main result establishes that the so-called *standard trisection*, determined by three equiangular inradius segments, minimizes the maximum relative diameter [3, Th. 3.5 and Prop. 5.1] for any set of that class. In addition, the optimal set for this problem is also characterized, up to dilations, as the intersection of the unit circle with a certain equilateral triangle [3, Th. 4.7]. We remark that, although the results in [3] are stated for divisions into three subsets of *equal areas*, all of them also hold in the case of unequal areas.

This paper is inspired precisely in these last two references involving the maximum relative diameter [11,3]. As a natural continuation, we shall focus on the class $\mathcal{C}_k$ of k-rotationally symmetric planar convex bodies (that is, compact convex sets) for each $k \in \mathbb{N}$, $k \geqslant 3$. And for each set in $\mathcal{C}_k$, we shall investigate the divisions into k connected subsets minimizing the maximum relative diameter functional. In this setting we shall consider two different types of divisions, namely k-*subdivisions* and k-*partitions*. For a given set $C \in \mathcal{C}_k$, a k-partition of C will be a decomposition of C into k connected subsets by k curves starting at different points of ∂C, and with all of them meeting at an interior point in C. On the other hand, a k-subdivision of C will be a general decomposition of C into k connected subsets, with no additional restrictions.

The goal of this paper is studying the minimizing k-partitions and minimizing k-subdivisions for the maximum relative diameter, for any k-rotationally symmetric planar convex body. In fact, we shall see that the so-called *standard k-partition* (consisting of k inradius segments symmetrically placed) is a minimizing