J. Math. Anal. Appl. 435 (2016) 917–943

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An inverse radiative coefficient problem arising in a two-dimensional heat conduction equation with a homogeneous Dirichlet boundary condition in a circular section [☆]

Liu Yang [a,b,*], Zui-Cha Deng [b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, People's Republic of China
[b] Department of Mathematics, Lanzhou Jiaotong University, Lanzhou, Gansu, 730070, People's Republic of China

ARTICLE INFO

Article history:
Received 25 September 2014
Available online 20 October 2015
Submitted by H. Kang

Keywords:
Convergence
Nonlinear inverse coefficient problem
Optimal control
Radiative coefficient
Singularity
Stability

ABSTRACT

In this study, we investigate the well-posedness of the solution of an optimal control problem related to a nonlinear inverse coefficient problem. Problems of this type have important applications in several fields of applied science. Unlike other terminal control problems, the observation data are only given for a fixed direction rather than for the whole domain, which may make the conjugate theory for parabolic equations ineffective. Moreover, the coefficients in our model are singular, so we propose some weighted Sobolev spaces. Based on the optimal control framework, the problem is transformed into an optimization problem and the existence of the minimizer is established. After deducing the necessary conditions that must be satisfied by the minimizer, we prove the uniqueness and stability of the minimizer. Following a minor modification of the cost functional and imposing some a priori regularity conditions on the forward operator, we obtain the convergence of the minimizer for the noisy input data considered in this study. The results obtained in this study are interesting and useful, and they can be extended to more general parabolic equations with singular coefficients.

1. Introduction

In this study, we investigate the existence, uniqueness, stability, and convergence of a solution to an optimal control problem that corresponds to a nonlinear inverse coefficient problem associated with two-dimensional heat conduction. The definite domain is of circular or sectorial type, so it is convenient to

☆ Supported by National Natural Science Foundation of China (Nos. 11061018, 11261029, and 11461039), NSF of Gansu Province of China (No. 145RJZA124), Youth Foundation of Lanzhou Jiaotong University (2011028), Long Yuan Young Creative Talents Support Program (No. 252003), and the Joint Funds of NSF of Gansu Province of China (No. 1212RJZA043).

* Corresponding author at: School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, People's Republic of China.

E-mail addresses: l_yang218@163.com (L. Yang), zc_deng78@hotmail.com (Z.-C. Deng).

http://dx.doi.org/10.1016/j.jmaa.2015.10.043

adopt the system of polar coordinates. In other words, the domain can be transformed into a rectangle. Our task is to identify the radiative coefficient by using the extra condition specified at an internal line. Inverse problems of this type have important applications in several fields of applied science. The problem can be stated in the following form.

Problem P. Consider an initial–boundary value problem of second order parabolic equation as follows

$$\begin{cases} u_t - \Delta u + q\left(\sqrt{x^2+y^2}\right) u = \tilde{f}(x,y,t), & (x,y) \in \tilde{\Omega}, \quad t \in (0,T], \\ u|_{\partial\tilde{\Omega}\times(0,T]} = 0, & \\ u|_{t=0} = \tilde{\varphi}(x,y), & (x,y) \in \tilde{\Omega}, \end{cases} \tag{1.1}$$

where $\tilde{\Omega} \subset \mathbb{R}^2$ is a circular sector. By using the polar coordinates transformation

$$r = \sqrt{x^2+y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

the equation (1.1) can be rewritten in the following form:

$$\begin{cases} u_t - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + q(r)u = f(r,\theta,t), & (r,\theta) \in \Omega, \quad t \in (0,T], \\ u|_{\theta=0} = u|_{\theta=\Theta} = u|_{r=R} = 0, & \\ \lim_{r\to 0^+} r\frac{\partial u}{\partial r} = 0, & \\ u|_{t=0} = \varphi(r,\theta), & \end{cases} \tag{1.2}$$

where $\Omega = (0,R) \times (0,\Theta)$, $f(r,\theta,t)$, and $\varphi(r,\theta)$ are given smooth functions

$$f(r,\theta,t) = \tilde{f}(r\cos\theta, r\sin\theta, t), \qquad \varphi(r,\theta) = \tilde{\varphi}(r\cos\theta, r\sin\theta),$$

and $q(r)$ is an unknown radiative coefficient that needs to be determined. Assume that an additional condition is given as follows:

$$u(r,\theta_0,T) = g(r), \tag{1.3}$$

where θ_0 is a fixed angle in $(0,\Theta)$ and $g(r)$ is a known function. We determine the functions u and q that satisfy (1.2)–(1.3).

In this study, we only consider the homogeneous Dirichlet boundary condition in (1.1).

The parameter q (in general, $q < 0$) arising in (1.1) is called the reproduction coefficient, which can be used to determine the critical size of the domain. In the case where $q > 0$, the mathematical model can be used to describe the heat transfer process. In addition to the heat source f, the zeroth-order term qu is also treated as a heat source, which depends on both the location and temperature. In fact, when $q > 0$, the decrease in the temperature is faster than the case where $q = 0$, which is similar to the thermal radiation phenomena in high temperature conditions. Thus, the parameter q is also referred to as the radiative coefficient in some previous studies, e.g., [31].

Inverse coefficient problems for parabolic equations have been studied comprehensively previously, see [2,9,17,22,24]. The inverse problem of identifying the radiative coefficient $q(x)$ in the following heat conduction equation

$$u_t - \Delta u + q(x)u = f(x,t), \qquad (x,t) \in Q,$$