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The eigenvalue ratio for a class of densities

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ABSTRACT

We investigate the nature of the eigenvalues for vibrating strings with the density function

$$\rho = \rho(x,t) = \begin{cases} -x & \text{if } -1 \le x \le 0 \\ tx & \text{if } 0 \le x \le 1 \end{cases}$$

where $t > 0$. The nth eigenvalue $\lambda_n(t)$ has a monotonicity property when t is changed. By means of Bessel functions, we obtain the limits of $\lambda_n(t)$ as $t \to 0$ and as $t \to \infty$. We also prove that the minimum of the ratio $\lambda_2(t)/\lambda_1(t)$ for $t > 0$ occurs at $t = 1$.

1. Introduction

If a string with a nonnegative integrable density $\rho(x)$, $x \in [-1,1]$, is fixed at the endpoints $x = -1$ and $x = 1$ under unit tension, then the natural frequencies of the string are determined by the eigenvalues of the boundary value problem

$$\begin{cases} u''(x) + \lambda\rho(x)u(x) = 0 \quad \text{in } (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{1.1}$$

In this paper, we investigate the nature of the eigenvalues of (1.1) for the density function

$$\rho = \rho(x,t) = \begin{cases} -x & \text{if } -1 \le x \le 0 \\ tx & \text{if } 0 \le x \le 1 \end{cases} \tag{1.2}$$

where $t > 0$. Indicating their dependence on the parameter t, we denote these eigenvalues by

$$0 < \lambda_1(t) < \lambda_2(t) < \lambda_3(t) < \cdots.$$

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It is useful to note that if $u_n(x,t)$ is the nth eigenfunction corresponding to $\lambda_n(t)$, then $u_n(-x,t)$ is the nth eigenfunction corresponding to $\lambda_n(1/t)$. Moreover,

$$\lambda_n(1/t) = t\lambda_n(t). \tag{1.3}$$

It follows that the ratio of the nth to the mth eigenvalues of a vibrating string with fixed endpoints and with density given by (1.2) satisfies

$$\frac{\lambda_n(1/t)}{\lambda_m(1/t)} = \frac{\lambda_n(t)}{\lambda_m(t)}. \tag{1.4}$$

The purpose of this paper is to prove the following results:

(I) For every fixed n, $\lambda_n(t)$ is a strictly decreasing function of t. Moreover,

$$\lim_{t\to\infty} \lambda_n(t) = 0 \qquad \text{and} \qquad \lim_{t\to 0} \lambda_n(t) = \alpha_n, \tag{1.5}$$

where α_n is the unique solution of the equation

$$\frac{J_{1/3}\left(\frac{2}{3}\sqrt{x}\right)}{J_{-1/3}\left(\frac{2}{3}\sqrt{x}\right)} + \left(\frac{x}{9}\right)^{1/3}\frac{\Gamma(2/3)}{\Gamma(4/3)} = 0 \tag{1.6}$$

in the interval $(\lambda_{2n-1}(1), \lambda_{2n}(1))$. Here J_p denotes the Bessel function of the first kind of order p, and Γ denotes the Gamma function (Theorem 3.3).

(II) The ratio $\lambda_2(t)/\lambda_1(t)$ attains its minimum at $t=1$. Hence

$$\frac{\lambda_2(t)}{\lambda_1(t)} \geq \frac{\lambda_2(1)}{\lambda_1(1)} \approx 2.41871$$

for all $t > 0$ (Theorem 4.1).

There has been much work devoted to finding bounds on the ratio of the first two eigenvalues (see [2,5–9] and references therein). It is a result of Huang [6] that if ρ is a symmetric single-well density, then

$$\lambda_2/\lambda_1 \leq 4 \tag{1.7}$$

with equality if and only if ρ is constant a.e. Horváth [5] gave a counterexample (step-function) which shows that the result (1.7) cannot be extended to any class of nonsymmetric single-well densities with fixed transition point q including $q = 0$, the midpoint of the interval $[-1, 1]$. An additional counterexample to this fact is given by the density (1.2). In fact, from (1.4) and (1.5), one has

$$\lim_{t\to\infty} \frac{\lambda_2(t)}{\lambda_1(t)} = \lim_{t\to 0} \frac{\lambda_2(t)}{\lambda_1(t)} = \frac{\alpha_2}{\alpha_1} \approx 5.4388 > 4.$$

Here we note that the equation (1.6) can be solved numerically for α_n with the results $\alpha_1 \approx 11.3502$, $\alpha_2 \approx 61.7324, \cdots$. Also, we remark that the density (1.2) is a continuous function.

2. Monotonicity properties of eigenvalues

Let $u_n(x,t)$ be the nth eigenfunction of (1.1)–(1.2) corresponding to $\lambda_n(t)$, normalized so that

$$\int_{-1}^{1} \rho(x,t)u_n^2(x,t)dx = 1. \tag{2.1}$$