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A Harnack inequality on the boundary of the unit ball

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ABSTRACT

In this paper we extend and simplify the main result of [4].

1. Introduction

Let $\mathbf{B}_n \subset \mathbb{C}^n$ be the unit ball and $\partial\mathbf{B}_n$ be the boundary of $\mathbf{B}_n$. A pluriharmonic function f is a function defined in $\mathbf{B}_n$ such that $f = g + \bar{h}$, where g and h are holomorphic functions in $\mathbf{B}_n$. A pluriharmonic mapping $f = (f_1, \ldots, f_M)$ is a function so that f_i $(i = 1, 2, \cdots, M)$, are pluriharmonic functions. A twice differentiable function f defined in an open subset Ω of the Euclidean space $\mathbf{R}^n$ will be called harmonic if it satisfies the following differential equation

$$\triangle f(x) = D_{11}f(x) + D_{22}f(x) + \cdots + D_{nn}f(x) = 0.$$

Every pluriharmonic mapping is harmonic and in complex dimension 1 the classes coincide.

For any $z = (z_1, \cdots, z_2)^t$ and $w = (w_1, \cdots, w_n)^t \in \mathbb{C}^n$, the inner product and the corresponding norm are given by $\langle z, w\rangle := \sum_{j=1}^{n} z_j\overline{w_j}$ and $\| z \| := \langle z, z\rangle^{\frac{1}{2}}$. Let $z_j = x_j + iy_j \in \mathbb{C}$ for $1 \le j \le n$, the real version of $z \in \mathbb{C}^n$ is bounded by $z' = (x_1, y_1, \cdots, x_n, y_n)^t \in \mathbf{R}^{2n}$.

Let $\mathbf{B}^n(x_0, R)$ be a closed ball in $\mathbf{R}^n$ with radius R and center at x_0. The Harnack's inequality (cf. [5]) states that, if f is continuous on $\mathbf{B}^n(x_0, R)$ and harmonic on its interior, then for any point x with $|x - x_0| = r < R$

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$$\frac{1-(1/R)}{[1+(r/R)]^{n-1}}f(x_0) \le f(x) \le \frac{1+(r/R)}{[1-(r/R)]^{n-1}}f(x_0). \tag{1}$$

For general domains $\Omega \subset \mathbf{R}^n$ the inequality can be stated as follows: if D is a bounded domain with $\overline{D} \subset \Omega$, then there is a constant C such that $f(x) \le Cf(y)$, $x, y \in D$, for every twice differentiable, harmonic and positive function f. The constant C is independent of f, it depends only on the domain D and Ω.

Let $\mathbf{B}^n$ be the unit ball in $\mathbf{R}^n$. For $x_0 \in \partial\mathbf{B}^n$, the tangent space $T_{x_0}(\partial\mathbf{B}^n)$ is defined by

$$T_{x_0}(\partial\mathbf{B}^n) := \{x \in \mathbf{R}^n : \langle x_0, x\rangle = 0\}.$$

Similarly, given a point $z_0 \in \partial\mathbf{B}_n \subset \mathbf{C}^n$, then $z' \in \partial\mathbf{B}^{2n}$. The tangent space $T_{z_0'}(\partial\mathbf{B}^{2n})$ can be defined similarly.

For $f : \mathbf{B}_n \mapsto \mathbf{B}_N$, denote $J_f(z_0')$ by the $2N \times 2n$ Jacobian matrix of f at z_0 in terms of real coordinates. For a bounded domain $V \in \mathbb{C}^n$, $C^\alpha(V)$ for $0 < \alpha < 1$ is the set of all functions f on V for which

$$\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} :\ z, w \in \overline{V}\right\}$$

is bounded. $C^{1+\alpha}$ is the set of all functions f on V whose kth order partial derivatives exist and belong to $C^\alpha(V)$ for integer $k \ge 0$.

In the paper [3] the authors established the boundary version of the classical Schwarz lemma for holomorphic functions in the unit ball B^n in $\mathbf{C}^n$ by proving the following: If $z_0 \in \partial\mathbf{B}_n$ is a fixed point of a holomorphic mapping $f : \mathbf{B}_n \to \mathbf{B}_n$, then there exists a positive eigenvalue λ of $J_f(z_0)$ such that the sharp estimate holds $\lambda \ge \frac{|1-\langle z_0,a\rangle|^2}{1-|a|^2}$, where $a = f(0)$. Moreover, $\overline{J_f(z_0)}^t z_0 = \lambda z_0$. In particular, if $n = 1$ and $a = 0$, the estimate reduces to $f'(1) \ge 1$, which is the well known boundary Schwarz lemma.

One may pose a question whether there exists similar result for harmonic or pluriharmonic mappings which send the unit ball onto itself. Very recently in the paper [4] Liu, Dai and Pan derived the above type result for pluriharmonic mappings which send the unit ball in $\mathbf{C}^n$ into the unit ball in $\mathbf{C}^m$. In fact, they proved the following Theorem A.

Theorem A. *Let $f : \mathbf{B}_n \mapsto \mathbf{B}_N$ be a pluriharmonic mapping for $n, N \ge 1$. If f is $C^{1+\alpha}$ at $z_0 \in \partial\mathbf{B}_n$ and $f(z_0) = w_0 \in \partial\mathbf{B}_N$, then we have*

(I) $J_f(z_0')\beta \in T_{w_0'}(\partial\mathbf{B}^{2N})$ for any $\beta \in T_{z_0'}(\partial\mathbf{B}^{2n})$;

(II) There exists a positive number $\lambda \in \mathbf{R}$ such that $J_f(z_0')^t w_0' = \lambda z_0'$, where z_0' and w_0' are real versions of z_0 and w_0 respectively, and $\lambda \ge \frac{1-\|f(0)\|}{2^{2n-1}} > 0$.

The proof of the above theorem in [4] is rather complicated and cannot be applied to the class of harmonic mappings. They use an approach from the paper [3].

We use the following property of harmonic mappings. A composition and a recomposition of a harmonic mapping with a unitary transformation is itself harmonic. For this property and other important properties of harmonic mappings we refer to the book [1] see also [2,6,7]. We extend and simplify the above theorem as follows.

Theorem 1. *Let f be a harmonic mapping of the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ onto $\mathbf{B}^m \subset \mathbf{R}^m$ having differentiable extension to the boundary point $a \in \partial\mathbf{B}_n$ such that $b = f(a) \in \partial\mathbf{B}_N$. Then $f'(a)^t b = \lambda a$. Moreover*

(i) $f'(a)(T_a(\partial\mathbf{B}^n)) \subset T_b(\partial\mathbf{B}^n)$;

(ii) There exists a positive number $\lambda \in \mathbf{R}$ such that $\lambda \ge \frac{1-|f(0)|}{2^{n-1}}$. Here $f'(a) = J_f(a)$.

Remark 1. As it will be seen on the proof, all the conclusions of this theorem except (*ii*) still hold if we remove the assumption that f is being harmonic.