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A note on the complexity function and entropy of pseudogroups

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ABSTRACT

We examine the connections between complexity of a pseudogroup, its equicontinuity, the mixing property and entropy. We prove that the entropy of a pseudogroup can be (under some additional assumptions) computed using a continuous and dynamically generating pseudometric.

1. Introduction

In [9], Ghys, Langevin and Walczak introduced the notion of entropy for foliations of Riemannian manifolds and finitely generated pseudogroups of local homeomorphisms. This concept applies also to laminations as observed in [7].

Pseudogroups of local homeomorphisms are natural generalizations of group actions on topological, in particular compact metric spaces (each group action generates a pseudogroup). Another important example is the holonomy pseudogroup of a foliated space defined by a regular covering by flow boxes. Therefore it is natural to ask about connections between dynamics of a pseudogroup and its entropy. This problem was studied by many authors (see for example [4,5,9,14,19,21,22]). Note that the value of the entropy of a pseudogroup depends both on the entropy of generators and the growth type of the pseudogroup. In particular, it depends on the choice of a generating set. But if the entropy is equal to zero for one generating set, then it vanishes for all of them.

We study the structure of pseudogroups with zero entropy. It has been conjectured that every distal pseudogroup has zero entropy. To our best knowledge this problem is still open, although some special cases have been answered positively: for example the entropy of a compact minimal distal foliated bundle was shown to vanish whenever its holonomy group has linear growth [4] or for the case of the foliations

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of codimension one [14]. Dynamical properties and entropy of pseudogroups were also considered in [11–13, 17,22].

A special class of zero entropy pseudogroups are equicontinuous pseudogroups (see for example [1,2,9, 15,18,20]). The authors of [1] generalized some properties of Riemannian foliations on closed manifolds to compact equicontinuous foliated spaces. For example, it is shown there that all holonomy covers of the leaves are quasi-isometric to each other. In [20] it is proved that equicontinuous transversely conformal foliations are Riemannian, which also follows from the main result of [2] in the case of dense leaves. In [9] it was shown that every equicontinuous pseudogroup has zero entropy. We prove that equicontinuity of a pseudogroup acting on a compact space implies its bounded complexity. We also study a connection between complexity of a pseudogroup and (a version of) the weak mixing property. This is an extension of [5], where the connection between dynamics of a system generated by a single map and its complexity is examined.

In the last part of the article we will show that entropy of a pseudogroup can be (under some additional assumptions) computed using a continuous and dynamically generating pseudometric. This result is known for the case of entropy of amenable groups and also was proved (in [16]) in the case of sofic entropy.

2. Terminology and notation

We assume that X is a compact metrizable space. Let $d\colon X \times X \to \mathbb{R}_+$ be a function. We say that d is a *pseudometric* if d is symmetric and fulfills the triangle inequality, that is, for all $x, y, z \in X$ one has $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$. We say that d is a *semimetric* if it is symmetric and definite ($d(x, y) = 0$ if and only if $x = y$).

Given a function $d\colon X \times X \to \mathbb{R}_+$ (not necessarily a semi- or pseudometric), a set $A \subset X$ and $x \in X$, recall the definitions

$$\operatorname{dist}_d(A, x) := \inf_{a\in A} d(a, x) \text{ and } \operatorname{diam}_d(A) := \sup_{a,b\in A} d(a, b).$$

Recall that a function $F\colon X \to \mathbb{R}$ is *lower semicontinuous* if for any $x \in X$ one has

$$\liminf_{y\to x} F(y) \geq F(x).$$

A lower semicontinuous function on a compact space attains its minimum and a pointwise supremum of lower semicontinuous functions is lower semicontinuous (see [6, Chapter 6.2]).

Let $\operatorname{Homeo}(X)$ denote the family of all homeomorphisms between open subsets of a topological space X. For a function g let D_g denote the domain of g and let Im_g denote its image. Given $g, h \in \operatorname{Homeo}(X)$, we denote by $g \circ h$ the restriction $g \circ h|_{h^{-1}(\operatorname{Im}_h \cap D_g)}$. Recall that a subfamily $\mathcal{G}$ of $\operatorname{Homeo}(X)$ is called a *pseudogroup* if the following conditions are satisfied:

1. $g \circ h \in \mathcal{G}$ for all $g, h \in \mathcal{G}$,
2. $g^{-1} \in \mathcal{G}$ for each $g \in \mathcal{G}$,
3. $g_{|U} \in \mathcal{G}$ for each $g \in \mathcal{G}$ and each open set $U \subset D_g$,
4. for all $g \in \operatorname{Homeo}(X)$ and for any open cover $\mathcal{U}$ of D_g, if $g_{|U} \in \mathcal{G}$ for each $U \in \mathcal{U}$, then $g \in \mathcal{G}$,
5. $\mathrm{id}_X \in \mathcal{G}$ (this is equivalent to the fact that $\bigcup\{D_g \mid g \in \mathcal{G}\} = X$).

We say that $G \subset \operatorname{Homeo}(X)$ generates $\mathcal{G}$ if $\mathcal{G}$ is the smallest (with respect to inclusion) pseudogroup containing G. Let G be a finite and symmetric set generating a pseudogroup $\mathcal{G}$. We say that G is *good* if for each $g \in G$ there is a compact set $K_g \subset D_g$ such that the family $\{g_{|\operatorname{int} K_g} \mid g \in G\}$ generates $\mathcal{G}$. A pseudogroup which admits a good generating set is called a *good pseudogroup*. Given a good pseudogroup