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The para-Racah polynomials

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ABSTRACT

New bispectral polynomials orthogonal on a quadratic bi-lattice are obtained from a truncation of Wilson polynomials. Recurrence relation and difference equation are provided. The recurrence coefficients can be encoded in a perturbed persymmetric Jacobi matrix. The orthogonality relation and an explicit expression in terms of hypergeometric functions are also given. Special cases and connections with the para-Krawtchouk polynomials and the dual-Hahn polynomials are also discussed.

1. Introduction

Hypergeometric orthogonal polynomials have numerous applications. We shall be concerned with polynomials that have $q = 1$ as base. The Askey tableau presents a hierarchical organization of these special functions [8]. It is comprised of a continuous part and of a discrete one. At the top of the continuous part are the Wilson polynomials expressed in terms of ${}_4F_3$ generalized hypergeometric series. A standard truncation condition on the parameters of the Wilson polynomials leads to the Racah polynomials which are orthogonal over a finite set of real points that form a quadratic lattice. The simplest limiting case of the Racah polynomials is that of the Krawtchouk polynomials orthogonal on the linear lattice.

Two of us have identified a family of orthogonal polynomials that fall outside the Askey scheme [10]. These para-Krawtchouk, as they were called, proved orthogonal over a linear bi-lattice formed by superimposing two linear lattices shifted one with respect to the other. The para-Krawtchouk polynomials naturally arise in quantum transport problems over spin chains [5,10].

We here identify the polynomials that are orthogonal with respect to quadratic bi-lattices. They are obtained from the Wilson polynomials through a novel truncation condition. They have the dual-Hahn polynomials as a special case and the para-Krawtchouk polynomials as a special limit.

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Consider the Wilson polynomials with parameters a, b, c, d denoted by $W_n(x^2; a, b, c, d)$. They obey the recurrence relation [8]

$$-(a^2 + x^2)\tilde{W}_n(x^2) = A_n\tilde{W}_{n+1}(x^2) - (A_n + C_n)\tilde{W}_n(x^2) + C_n\tilde{W}_{n-1}(x^2) \tag{1.1}$$

where

$$\tilde{W}_n(x^2) = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} \tag{1.2}$$

with $(a)_k = a(a+1)\dots(a+k-1)$ the usual Pochhammer symbol and

$$A_n = \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \tag{1.3}$$

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \tag{1.4}$$

They also satisfy the difference equation

$$n(n+a+b+c+d-1)\tilde{W}_n(x^2) = \overline{\mathcal{D}(x)}\tilde{W}_n((x+i)^2) - (\overline{\mathcal{D}(x)} + \mathcal{D}(x))\tilde{W}_n(x^2) + \mathcal{D}(x)\tilde{W}_n((x-i)^2), \tag{1.5}$$

where $\overline{\mathcal{D}(x)}$ is the complex conjugate of $\mathcal{D}(x)$

$$\mathcal{D}(x) = \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{(2ix)(2ix+1)}. \tag{1.6}$$

The Wilson polynomials with parameters a, b, c, d, admit an explicit expression given by

$$\tilde{W}_n(x^2; a, b, c, d) = {}_4F_3\left[\begin{matrix}-n,\ n+a+b+c+d-1,\ a-ix,\ a+ix \\ a+b,\ a+c,\ a+d\end{matrix}; 1\right] \equiv \sum_k A_{n,k}\Phi_k(x^2) \tag{1.7}$$

where

$$A_{n,k} = \frac{(-n)_k(n+a+b+c+d-1)_k}{(1)_k(a+b)_k(a+c)_k(a+d)_k}, \quad \Phi_k(x^2) = (a-ix)_k(a+ix)_k. \tag{1.8}$$

It is well-known [8] that the Wilson polynomials can be reduced to a finite set of $N+1$ orthogonal polynomials if

$$A_N C_{N+1} = 0. \tag{1.9}$$

This can be achieved by setting $a+b$, $a+c$, $a+d$, $b+c$, $b+d$ or $c+d$ equal to $-N$. This leads to the Racah polynomials. Another possibility is to take $a+b+c+d-1 = -N$, but this introduces a singularity in the denominator of the recurrence coefficients. However, one can get around this problem with the use of limits and obtain new orthogonal polynomials. Our goal here is to study and characterize these polynomials which we shall call para-Racah polynomials.

This new truncation of Wilson polynomials will be presented in section 2 for odd values of N and the corresponding recurrence relation and difference equation will be obtained. In section 3, we derive an explicit expression for the para-Racah polynomials in terms of hypergeometric functions and compute their weights. The para-Racah polynomials with N even are presented in section 4. In section 5, we discuss special cases and the connections with the para-Krawtchouk polynomials and the dual-Hahn polynomials.