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Embedding of generalized Lipschitz classes into classes of functions with Λ-bounded variation ☆

Heping Wang

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

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ABSTRACT

In this paper, sufficient and necessary conditions for embedding of generalized Lipschitz classes H_p^ω, $1 < p < \infty$ into classes ΛBV of functions with Λ-bounded variation are obtained under the mild restriction on ω.

1. Introduction and results

Jordan's classical concept of bounded variation was generalized by many authors and in various ways (see [1]). In particular, Waterman had introduced the notion of a function of Λ-bounded variation (see [18–20]).

Let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\sum 1/\lambda_n$ diverges. We say that a real-valued 1-periodic function f on $[0, 1]$ is of Λ-bounded variation (ΛBV) if

$$V_\Lambda(f) := \sup_{\mathcal{I}} \sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < +\infty,$$

where the supremum is taken over all sequences $\mathcal{I} = \{I_n\} = \{[a_n, b_n]\}$ of non-overlapping intervals in $[0, 1]$, $f(I_n) = f(b_n) - f(a_n)$. It is obvious that if all $\lambda_n = 1$, $\{1\}$BV coincides with the class BV of functions of the usual bounded variation.

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E-mail address: wanghp@cnu.edu.cn.

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Let $\omega(t)$ be a modulus of continuity, i.e., a continuous, subadditive, and nondecreasing function on $[0,+\infty)$ satisfying $\omega(0)=0$. For $1 \le p \le \infty$, denote by $H_p^\omega \equiv H_p^{\omega(t)}$ the class of 1-periodic functions for which $\|f\|_{H_p^\omega} := \|f\|_p + \sup\limits_{t>0} \frac{\omega(f;t)_p}{\omega(t)} < \infty$, where

$$\omega(f;t)_p := \begin{cases} \sup_{0\le h\le t} \left\{ \int_0^1 |f(x+h)-f(x)|^p dx \right\}^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{0\le h\le t} \sup_{x\in[0,1]} |f(x+h)-f(x)|, & p=\infty \end{cases}$$

is the L_p modulus of continuity of f. We write H^ω instead of H_∞^ω and H_p^α $(0<\alpha\le 1)$ instead of $H_p^{t^\alpha} \equiv \mathrm{Lip}(\alpha,p)$, the Lipschitz class, for brevity. It is well known that for each modulus of continuity ω there exists a concave modulus of continuity ω^* such that $\omega(t) \le \omega^*(t) \le 2\omega(t)$ for $t\in[0,\infty)$. Then $H_p^\omega = H_p^{\omega^*}$. In what follows, we always assume $0<\alpha\le 1$.

In recent years, much attention is drawn on the relationship of the class $\Lambda\mathrm{BV}$ and the Lipschitz class H_p^ω. Sharp estimates of the L_p-modulus of continuity $(1\le p<\infty)$ of a function in terms of its Λ-variation were obtained in [14,5,6]. Furthermore, Goginava gave the necessary and sufficient condition for the inclusion of the class $\Lambda\mathrm{BV}$ in the class H_p^ω $(1\le p<\infty)$ (see [2]):

$$\Lambda\mathrm{BV} \subset H_p^\omega \quad \text{if and only if} \quad \max_{1\le m\le n} \frac{m^{1/p}}{\sum_{k=1}^m 1/\lambda_k} = O\left(n^{1/p}\omega\left(\frac{1}{n}\right)\right) \quad \text{as } n\to\infty.$$

This result was later generalized in [4,3,17,16].

On the other hand, the reverse embedding is also investigated. For $p=\infty$, Medvedeva showed in [10] that a sufficient and necessary condition for the embedding $H^\omega \subset \Lambda\mathrm{BV}$ is that $\sum_{k=1}^\infty \frac{\omega(t_k)}{\lambda_k} < \infty$ for any sequence $\{t_k\}_{k=1}^\infty$, $t_k\ge 0$, $\sum_{k=1}^\infty t_k \le 1$. The present author studied the embedding $H_p^\omega \subset \Lambda\mathrm{BV}$ for $1\le p<\infty$ in [15], and showed that a sufficient and necessary condition for the embedding $H_p^\alpha \subset \{n^\beta\}\mathrm{BV}$, $1<p<\infty$ is $\alpha > \max\{1/p, 1-\beta\}$. Furthermore, Lind proved in [7] that for $1<p<\infty$, $1/p<\alpha<1$, the embedding $H_p^\alpha \subset \Lambda\mathrm{BV}$ holds if and only if

$$\sum_{n=0}^\infty \Big(\sum_{k=2^n}^{2^{n+1}} \Big(\frac{1}{k^{\alpha-1/p}\lambda_k} \Big)^{p'} \Big)^{r'/p'} < \infty, \tag{1.1}$$

where $p' = \frac{p}{p-1}$, $r = \frac{1}{\alpha-1/p}$, and $r' = \frac{1}{1+1/p-\alpha}$. We remark that (1.1) cannot be extended directly to H_p^ω for general modulus ω.

This paper is devoted to investigating the embedding $H_p^\omega \subset \Lambda\mathrm{BV}$, $1<p<\infty$ under some mild restriction on ω. The condition $\omega(t) = O(t^{1/p})$ is necessary for the embedding (see [15]). We say that a nonnegative function φ on $[0,\infty)$ satisfies Condition (P) if there exist positive constants γ, C_1, C_2 and $\tau\in(0,1)$ independent of $t>0$ such that for any $0<t_1\le t_2$,

$$\varphi(t_1)t_1^{-\gamma} \le C_1\,\varphi(t_2)t_2^{-\gamma} \quad \text{and} \quad \varphi(t_1)t_1^{\tau-1} \ge C_2\,\varphi(t_2)t_2^{\tau-1}$$

hold. We suppose that the functions $\omega(t)$ and $\omega_1(t) = t^{-1/p}\omega(t)$ satisfy Condition (P). This restriction on ω is justified so that functions in H_p^ω can be modified on a set of zero measure to be continuous, see [13]. A prototype of functions of ω is $\omega(t) = t^{\delta+1/p}(1+(\ln 1/t)_+)^\beta$, $0<\delta<1-1/p$, $\beta\in\mathbb{R}$, where $a_+=a$ if $a\ge 0$ and $a_+=0$ if $a<0$. We shall show in Section 3 that the above ω_1 is comparable with a concave modulus of continuity $\tilde{\omega_1}$.

Let $\tilde{\omega_1}^{-1}(t) := \sup\{x \mid \tilde{\omega_1}(x)\le t,\ x\in[0,\infty)\}$ be the right inverse function on $[0,\infty)$ of $\tilde{\omega_1}$. Then $\tilde{\omega_1}^{-1}$ is convex and increasing on $[0,\infty)$ with $\tilde{\omega_1}^{-1}(0)=0$. Let ψ be the complementary function of $\tilde{\omega_1}^{-1}$, i.e., $\psi(x) = \sup\limits_{y>0}\{|x|y - \tilde{\omega_1}^{-1}(y)\}$, $x\in\mathbb{R}$. Then $\psi(x)$ is also convex and increasing on $[0,\infty)$ with $\psi(0)=0$.