J. Math. Anal. Appl. 438 (2016) 720–737

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An abstract theorem in nonlinear analysis and two applications

A.C. Lazer [a], P.J. McKenna [b,*], R.H. Pellico [c]

[a] Department of Mathematics, University of Miami, Coral Gables, FL 33124, United States
[b] Department of Mathematics, University of Connecticut, Storrs, CT 06269, United States
[c] Department of Mathematics, Trinity College, Hartford, CT 06106, United States

ARTICLE INFO

Article history:
Received 20 November 2015
Available online 22 January 2016
Submitted by Goong Chen

Keywords:
Elliptic
Differential equation
Boundary value problem
Nonlinear

ABSTRACT

We prove abstract multiplicity theorems for a class of nonlinear equations on the Hilbert space $L^2(\Omega)$. Then we use these theorems to deduce new results for two different types of nonlinear boundary value problems.

1. Introduction

This paper is based on a long series of results on the nonlinear boundary value problem

$$\begin{aligned} \Delta u + bu^+ &= s\phi_1(x) \qquad x \in \Omega \\ u &= 0 \qquad x \in \partial\Omega \end{aligned} \tag{1}$$

where Ω is a bounded region in $\mathbb{R}^n$ and ϕ_1 denotes the first eigenfunction of the Laplacian with Dirichlet boundary conditions. The real number λ_1 is the first of the infinite sequence of eigenvalues $\lambda_n \to +\infty$. The nonlinearity bu^+ is a model nonlinearity for a nonlinearity of the form $f(u)$ where $f'(+\infty) = b$ and $f'(-\infty) = 0$. More generally, we can take $f'(-\infty) = a$ if we replace the bu^+ by $bu^+ - au^-$.

Since $\phi_1(x) \geq 0$, it is an elementary calculation that if the real number $s > 0$, the equation (1) has two obvious solutions if $\lambda_1 < b < +\infty$. These are given by $u_1 = -s\phi_1/\lambda_1 (\leq 0)$ and $u_2 = s\phi_1/(b - \lambda_1)(\geq 0)$. If on the other hand, $s = 0$, then $u \equiv 0$ is the unique solution, whereas if $s < 0$, there is no solution.

Generally, the literature can be summarized by the following two statements: if the interaction of the nonlinearity with the spectrum of the Laplacian is small, by which we mean $\lambda_1 < b < \lambda_2$, then the two

* Corresponding author.
E-mail addresses: a.lazer@math.miami.edu (A.C. Lazer), mckenna@math.uconn.edu (P.J. McKenna), ryan.pellico@trincoll.edu (R.H. Pellico).

http://dx.doi.org/10.1016/j.jmaa.2015.11.084
0022-247X/

"obvious" solutions are the only ones. On the other hand if there is more interaction with the spectrum, that is, if $b > \lambda_2$, then more "non-obvious" solutions appear. Let us make this more precise.

The earliest results go back to the early seventies [1], where the existence of at most two solutions was proved for more general nonlinearities and right hand sides. The existence of more solutions began in the early eighties in [6,7,10,12,14,21] when it was shown that if $b > \lambda_2$, there are at least four solutions. Much more recently, it was shown in [17,18], that as $b \to +\infty$, the number of solutions goes to plus infinity.

Remark 1.1. We should also mention the "superlinear" version of equation (1), namely

$$\begin{aligned} \Delta u + b(u^+)^p = s\phi_1(x) \qquad & x \in \Omega \\ u = 0 \qquad & \in \partial\Omega \end{aligned} \tag{2}$$

with $p > 1$. This is a model for the case where $f'(+\infty) = +\infty$. The natural conjecture here was that as the real number $s \to +\infty$, the number of solutions should also go to infinity. This proved difficult, initially, as it was only clear that there are at least two solutions. After an initial computer-aided proof of at least four solutions in [3], this was proved in [9], with further developments in [8,20].

Remark 1.2. For the ordinary differential equation version of equation (1), not surprisingly, more is known. In one dimension, we take $\Omega = (0, \pi)$ and then $\phi_1 = \sin(x)$ and $\lambda_n = n^2$. (Of course, there is also the corresponding superlinear problem.) In [13], it was shown that if $n^2 < b < (n+1)^2$, then the one-dimensional version of equation (1) has at least $2n$ solutions. In [5], it was shown that this is precise. In the case of the superlinear version, the above-mentioned conjecture was proved, in [15] and in the case of radially symmetric solutions on a ball, in [4].

The plan of this paper is the following; first we shall prove our two abstract theorems for a semilinear operator equation similar to equation (1). The hypotheses on the linear part L will be outlined at the beginning of the next section. In the following sections, we shall give applications to two cases, the first a symmetry-breaking result related to a recent result of Pacella and Srikanth [19], and the second to a different type of nonlocal linear operator, namely the fractional Laplacian.

2. The abstract result

For the rest of this section, we will be working in a closed subspace H of the Hilbert space $L^2(\Omega)$, where Ω is a bounded region in $\mathbb{R}^n$. The unbounded selfadjoint linear operator $L : D(L) \to H$ will satisfy the following **hypotheses**

1. $-L$ has an infinite sequence of eigenvalues λ_n with $0 < \lambda_1 < \lambda_2 < \lambda_3 \le \lambda_4 \le \cdots \le \lambda_n \le \ldots$ so that $-L\phi_n = \lambda_n\phi_n$.
2. The associated eigenfunctions ϕ_n are an orthonormal basis for H with $\phi_1(x) > 0$ for all $x \in \Omega$.
3. There exists $\varepsilon > 0$ such that $\phi_1(x) > \varepsilon\|\phi_2(x)\|, \forall x \in \Omega$.

Typically, the space H will either be all of $L^2(\Omega)$, or a subspace defined by certain symmetries. We will also assume that the map $u \to bu^+$ leaves H invariant, which will usually be satisfied if the symmetries are *even*.

Remark 2.1. The reader will immediately notice that the Laplacian, with Dirichlet boundary conditions of equation (1) satisfies these hypotheses. Of course, they apply to a much wider class than just this operator.