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The boundary quotient for algebraic dynamical systems

Nathan Brownlowe [a], Nicolai Stammeier [b,*,1]

[a] School of Mathematics and Applied Statistics, University of Wollongong, Australia
[b] Department of Mathematics, University of Oslo, Norway

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ABSTRACT

We introduce the notion of accurate foundation sets and the accurate refinement property for right LCM semigroups. For right LCM semigroups with this property, we derive a more explicit presentation of the boundary quotient. In the context of algebraic dynamical systems, we also analyse finiteness properties of foundation sets which lead us to a very concrete presentation. Based on Starling's recent work, we provide sharp conditions on certain algebraic dynamical systems for pure infiniteness and simplicity of their boundary quotient.

0. Introduction

All semigroups in this paper are assumed to be countable, discrete and left cancellative. Recall from [6] that a semigroup is *right LCM* if the intersection of two principal right ideals is either empty or another principal right ideal. Examples of right LCM semigroups come from *algebraic dynamical systems* (G, P, θ), which consist of an action θ of a right LCM semigroup P with identity by injective endomorphisms of a group G, subject to the condition that $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$ for all $p, q, r \in P$, see [5] for details and examples. It has been observed that the C^*-algebra $\mathcal{A}[G, P, \theta]$ associated to (G, P, θ) in [5] is isomorphic to the full semigroup C^*-algebra of the right LCM semigroup $G \rtimes_\theta P$, see [5, Theorem 4.4]. It is also known to be isomorphic to a Nica–Toeplitz algebra for a product system of right-Hilbert bimodules over the right LCM semigroup P, see [5, Theorem 7.9]. These two ways of viewing $\mathcal{A}[G, P, \theta]$ both indicate that this C^*-algebra tends to have proper ideals. Therefore, it is natural to search for a notion of a minimal quotient that is simple under reasonable assumptions on (G, P, θ).

With regard to C^*-algebras of product systems of right-Hilbert bimodules, this quotient ought to be a Cuntz–Nica–Pimsner algebra. But so far only Nica covariance has been defined for product systems over right LCM semigroups, see [5, Definition 6.4]. Even worse, it does not seem to be clear what the general

* Corresponding author.
E-mail addresses: nathanb@uow.edu.au (N. Brownlowe), nicolsta@math.uio.no (N. Stammeier).
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notion of Cuntz–Pimsner covariance for product systems over quasi-lattice ordered pairs should be, compare [11] and [20]. Recently, definitions for Cuntz–Pimsner covariance for product systems over Ore semigroups have been proposed in [13] and [1] which might lead to substantial progress in this direction. However, we remark that a right LCM semigroup can be far from satisfying the Ore condition.

There has been a successful attempt to identify the analogous quotient, called the *boundary quotient*, for full semigroup C^*-algebras of right LCM semigroups with identity, see [6]. In fact, the authors also indicate how one could define this object for general semigroups, see [6, Remark 5.5]. Let us briefly review the idea behind this quotient, which goes back to [7]: Firstly, recall from [4, Lemma 3.3] that the family of constructible right ideals $\mathcal{J}(S)$ for a right LCM semigroup with identity S consists only of $\emptyset$ and the principal right ideals in S. A finite subset F of S is called a *foundation set* if for every $s \in S$ there is $f \in F$ such that $sS \cap fS \neq \emptyset$. The boundary quotient $\mathcal{Q}(S)$ of $C^*(S)$ is then obtained by imposing the additional relation $\prod_{s\in F}(1 - e_{sS}) = 0$ for every foundation set F. It was shown in [6] that $\mathcal{Q}(S)$ recovers classical objects such as $\mathcal{O}_n$, provides an appealing perspective on Toeplitz and Cuntz–Pimsner algebras associated to self-similar actions, see [6, Subsection 6.4], and may yield plenty of interesting new C^*-algebras related to Zappa–Szép products of monoids which had not been considered before.

As we know that $G \rtimes_\theta P$ is right LCM for each algebraic dynamical system (G, P, θ), the boundary quotient $\mathcal{Q}(G \rtimes_\theta P)$ deserves a closer examination. As it turns out, for most standard examples of such dynamics, the resulting right LCM semigroup $S = G \rtimes_\theta P$ has two additional features: There are plenty of foundation sets F such that f_1S and f_2S are disjoint for all distinct $f_1, f_2 \in F$. Such finite subsets F will be called *accurate foundation sets*. More importantly, every foundation set F can be refined to an accurate foundation set F_a in the sense that for every $f_a \in F_a$ there is $f \in F$ such that $f_a \in fS$. This feature will be named the *accurate refinement property*, or property (AR) for short. If a right LCM semigroup S has property (AR), then the defining relation

$$\prod_{f\in F}(1 - e_{fS}) = 0 \quad \text{for every foundation set } F$$

can be replaced by the more familiar-looking relation

$$\sum_{f\in F_a} e_{fS} = 1 \quad \text{for every accurate foundation set } F_a,$$

see Proposition 2.4. We show that property (AR) is enjoyed by various types of known right LCM semigroups.

If we are given additional information on S in the sense that $S = G \rtimes_\theta P$ for a (nontrivial) algebraic dynamical system (G, P, θ), then we can say more about the structure of (accurate) foundation sets and hence about property (AR). This is the aim of Section 3, where we present a useful sufficient criterion on (G, P, θ) for $G \rtimes_\theta P$ to have property (AR), see Proposition 3.9. As an application, we show that $G \rtimes_\theta P$ has property (AR) provided that P is directed or that incomparable elements in P have disjoint principal right ideals, where we use $p \geq q :\Leftrightarrow p \in qP$, see Corollary 3.11. We note that these two options include the cases where P is a group, an abelian semigroup, a free semigroup, or a Zappa–Szép product $X^* \bowtie G$ for some self-similar action (G, X) as in [6]. In particular, the semigroups $G \rtimes_\theta P$ arising from irreversible algebraic dynamical systems as defined in [21] have property (AR). To achieve Proposition 3.9 and hence the aforementioned results, we use a celebrated lemma of B.H. Neumann from [17] about finite covers of groups by cosets of subgroups to conclude that it suffices to consider (accurate) foundation sets F for $G \rtimes_\theta P$ such that the index of $\theta_p(G)$ of G is finite for all $(g, p) \in F$, see Proposition 3.5.

Let (G, P, θ) satisfy the assumptions of Proposition 3.9, so that $G \rtimes_\theta P$ has property (AR). If we combine the alternative presentation for $\mathcal{Q}(G \rtimes_\theta P)$ obtained in Proposition 2.4 with the dynamic description $\mathcal{A}[G, P, \theta]$ of $C^*(G \rtimes_\theta P)$, we arrive at a presentation of $\mathcal{Q}(G \rtimes_\theta P)$ which emphasises that it originates from a dynamical system, see Corollary 4.1. However, we observe that $\mathcal{Q}(G \rtimes_\theta P)$ may fail to admit a natural