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Riemann integrability versus weak continuity ☆

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ABSTRACT

In this paper we focus on the relation between Riemann integrability and weak continuity. A Banach space X is said to have the weak Lebesgue property if every Riemann integrable function from $[0, 1]$ into X is weakly continuous almost everywhere. We prove that the weak Lebesgue property is stable under ℓ_1-sums and obtain new examples of Banach spaces with and without this property. Furthermore, we characterize Dunford–Pettis operators in terms of Riemann integrability and provide a quantitative result about the size of the set of τ-continuous nonRiemann integrable functions, with τ a locally convex topology weaker than the norm topology.

1. Introduction

The study of the relation between Riemann integrability and continuity on Banach spaces started in 1927, when Graves showed in [13] the existence of a vector-valued Riemann integrable function not continuous almost everywhere (a.e. for short). Thus, the following problem arises:

Given a Banach space X, determine necessary and sufficient conditions for the Riemann integrability of a function $f : [0, 1] \to X$.

A Banach space X for which every Riemann integrable function $f : [0, 1] \to X$ is continuous a.e. is said to have the Lebesgue property (LP for short). All classical infinite-dimensional Banach spaces except ℓ_1 do not have the LP. For more details on this topic, we refer the reader to [12,6,24,14,19].

Regarding weak continuity, Alexiewicz and Orlicz constructed in 1951 a Riemann integrable function which is not weakly continuous a.e. [2]. A Banach space X is said to have the weak Lebesgue property

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(WLP for short) if every Riemann integrable function $f : [0, 1] \to X$ is weakly continuous a.e. This property was introduced in [27]. Every Banach space with separable dual has the WLP and the example of [2] shows that $\mathcal{C}([0, 1])$ does not have the WLP. Other spaces with the WLP, such as $L^1([0, 1])$, can be found in [5] and [28]. In this paper we focus on the relation between Riemann integrability and weak continuity. In Section 2 we present new results on the WLP. In particular, we prove that the James tree space JT does not have the WLP (Theorem 2.4) and we study when $\ell_p(\Gamma)$ and $c_0(\Gamma)$ have the WLP in the non-separable case (Theorem 2.10). Moreover, we prove that the WLP is stable under ℓ_1-sums (Theorem 2.15) and we apply this result to obtain that $\mathcal{C}(K)^*$ has the WLP for every compact space K in the class MS (Corollary 2.18).

Alexiewicz and Orlicz also provided in their paper an example of a weakly continuous nonRiemann integrable function. V. Kadets proved in [15] that a Banach space X has the Schur property if and only if every weakly continuous function $f : [0, 1] \to X$ is Riemann integrable. Wang and Yang extended this result in [29] to arbitrary locally convex topologies weaker than the norm topology. In the last section of this paper we give an operator theoretic form of these results that, in particular, provides a positive answer to a question posed by Sofi in [26].

1.1. Terminology and preliminaries

All Banach spaces are assumed to be real. In what follows, X^* denotes the dual of a Banach space X. The weak and weak* topologies of X and X^* will be denoted by ω and ω^* respectively. By an operator we mean a linear continuous mapping between Banach spaces. The Lebesgue measure in $\mathbb{R}$ is denoted by μ. The interior of an interval I will be denoted by $\mathrm{Int}(I)$. The density character $\mathrm{dens}(T)$ of a topological space T is the minimal cardinality of a dense subset. A partition of the interval $[a, b] \subset \mathbb{R}$ is a finite collection of non-overlapping closed subintervals covering $[a, b]$. A tagged partition of the interval $[a, b]$ is a partition $\{[t_{i-1}, t_i] : 1 \le i \le N\}$ of $[a, b]$ together with a set of points $\{s_i : 1 \le i \le N\}$ that satisfy $s_i \in (t_{i-1}, t_i)$ for each i.

Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \le i \le N\}$ be a tagged partition of $[a, b]$. For every function $f : [a, b] \to X$, we denote by $f(\mathcal{P})$ the Riemann sum $\sum_{i=1}^{N} (t_i - t_{i-1}) f(s_i)$. The norm of $\mathcal{P}$ is $\|\mathcal{P}\| := \max\{t_i - t_{i-1} : 1 \le i \le N\}$. We say that a function $f : [a, b] \to X$ is Riemann integrable, with integral $x \in X$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ for all tagged partitions $\mathcal{P}$ of $[a, b]$ with norm less than δ. In this case, we write $x = \int_a^b f(t)dt$.

The following criterion will be used for proving the existence of the Riemann integral of certain functions:

Theorem 1.1. *(See [12].) Let $f : [0, 1] \to X$. The following statements are equivalent:*

1. *The function f is Riemann integrable.*
2. *For each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[0, 1]$ with $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0, 1]$ that have the same intervals as $\mathcal{P}_\varepsilon$.*
3. *There is $x \in X$ such that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[0, 1]$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[0, 1]$ with the same intervals as $\mathcal{P}_\varepsilon$.*

We will also be concerned about cardinality. Throughout this paper, $\mathfrak{c}$ denotes the cardinality of the continuum and $\mathrm{cov}(\mathcal{M})$ denotes the smallest cardinal such that there exist $\mathrm{cov}(\mathcal{M})$ nowhere dense sets in $[0, 1]$ whose union is the interval $[0, 1]$. This cardinal coincides with the smallest cardinal such that there exist $\mathrm{cov}(\mathcal{M})$ closed sets in $[0, 1]$ with Lebesgue measure zero whose union does not have Lebesgue measure zero (see [4, Theorem 2.6.14]).