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Accurate approximations for the complete elliptic integral of the second kind

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ABSTRACT

In this paper, we prove that the double inequality

$$\lambda S_{11/4,7/4}(1,r') < \mathcal{E}(r) < \mu S_{11/4,7/4}(1,r')$$

holds for all $r \in (0,1)$ if and only if $\lambda \le \pi/2 = 1.570796\cdots$ and $\mu \ge 11/7 = 1.571428\cdots$, where $r' = (1-r^2)^{1/2}$, $\mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt$ is the complete elliptic integral of the second kind, and $S_{p,q}(a,b) = [q(a^p - b^p)/(p(a^q - b^q))]^{1/(p-q)}$ is the Stolarsky mean of a and b.

1. Introduction

For $r \in (0,1)$, Legendre's complete elliptic integrals [1] of the first kind and the second kind are given by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}},$$

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2(t)}dt,$$

respectively. It is well known that

$$\mathcal{K}(0^+) = \mathcal{E}(0^+) = \frac{\pi}{2}, \quad \mathcal{K}(1^-) = \infty, \quad \mathcal{E}(1^-) = 1. \tag{1.1}$$

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They are the particular cases of the Gaussian hypergeometric function

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(-1<x<1),$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function. Indeed, we have

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n},\tag{1.2}$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}.\tag{1.3}$$

The complete elliptic integrals and Gaussian hypergeometric function have important applications in quasiconformal mappings, number theory, and other fields of the mathematical and mathematical physics. For instance, the Gaussian arithmetic–geometric mean AGM and the modulus of the plane Grötzsch ring can be expressed in terms of the complete elliptic integral of the first kind, and the complete elliptic integral of the second kind gives the formula of the perimeter of an ellipse. Moreover, Ramanujan modular equation and continued fraction in number theory are both related to the Gaussian hypergeometric function $F(a,b;c;x)$. In particular, many remarkable inequalities and properties for the complete elliptic integrals and Gaussian hypergeometric function can be found in the literature [2–6,8,11,13,14,17].

Recently, the bounds for the complete elliptic integral of the second kind $\mathcal{E}(r)$ have attracted the attention of many researchers. In [15], Vuorinen conjectured that the inequality

$$\mathcal{E}(r)\geq\frac{\pi}{2}M_{3/2}(1,r')\tag{1.4}$$

holds for all $r\in(0,1)$, where and in what follows $r'=(1-r^2)^{1/2}$, $M_p(a,b)$ is the pth power mean of a and b which is given by

$$M_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}\quad(p\neq0),\qquad M_0(a,b)=\sqrt{ab}.$$

Inequality (1.4) was proved by Barnard et al. in [6]. In [18, Theorem 2.4] and [16, Corollary 3.1], the authors proved that the double inequalities

$$\frac{23A(1,r')-5H(1,r')-2M_2(1,r')}{16}<\frac{2}{\pi}\mathcal{E}(r)<\frac{(24-5\sqrt{2}\pi)A(1,r')-(8-\pi-\sqrt{2}\pi)H(1,r')-(16-5\pi)M_2(1,r')}{2\pi(3-2\sqrt{2})}\tag{1.5}$$

and

$$\frac{(9r'^2+14r'+9)^2}{128(1+r')^3}<\frac{2}{\pi}\mathcal{E}(r)<\frac{\sqrt{4r'^2+(\pi^2-8)r'+4}}{\pi}\tag{1.6}$$

hold for all $r\in(0,1)$, where $A(a,b)=M_1(a,b)=(a+b)/2$ and $H(a,b)=M_{-1}(a,b)=2ab/(a+b)$ are respectively the arithmetic and harmonic means of a and b.