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Symmetry results for positive solutions of mixed integro-differential equations

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ABSTRACT

In this paper, we study symmetry property for positive solutions of mixed integro-differential equations

$$\begin{cases} (-\Delta)_x^{\alpha_1} u + (-\Delta)_y^{\alpha_2} u = f(u) & \text{in } B_1^N(0) \times B_1^M(0), \\ u = 0 & \text{in } (\mathbb{R}^N \times \mathbb{R}^M) \setminus (B_1^N(0) \times B_1^M(0)), \end{cases} \tag{0.1}$$

where $N,\ M \geq 1$, $x \in B_1^N(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, $y \in B_1^M(0) = \{y \in \mathbb{R}^M : |y| < 1\}$, the operator $(-\Delta)_x^{\alpha_1}$ denotes the fractional Laplacian of exponent $\alpha_1 \in (0,1)$ with respect to x, $(-\Delta)_y^{\alpha_2}$ denotes the fractional Laplacian of exponent $\alpha_2 \in (0,1)$ with respect to y. We make use of the Maximum Principle for small domain to start the moving planes to obtain the symmetry results for positive solutions.

1. Introduction

The study of radial symmetry of positive solutions to semilinear elliptic equations in bounded domains has been the concern of numerous authors along the last several decades. It was the seminal work by Gidas, Ni and Nirenberg [10] that settled this property of positive C^2-solutions for elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad B_1, \\ u = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{1.1}$$

They proved that any positive C^2-solution of (1.1) is radially symmetric and decreasing by the method of moving planes as in [20]. More later, Berestycki and Nirenberg in [3] gave a more simple proof of this result

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using a very powerful nonlinear strategy, the method of moving planes based on the Maximum Principle for small domain which is derived by the Aleksandrov–Bakelman–Pucci (ABP) estimate. More generally, if the domain is symmetric and convex with respect to a hyperplane then the solutions have the same symmetry. Related results in the whole space and exterior domains were obtained by Li [12], Reichel [15] and Sirakov [23], under the supplementary hypothesis that f is nonincreasing in a right neighborhood of zero.

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great applications and by important advances on the theory of nonlinear partial differential equations, see [4,6,9,14,16,21,22,24] for details. In some recent works, Guillen and Schwab in [11] proved an ABP estimate for integro-differential equations, Ros-Oton et al. obtained the Pohozaev identities in [19] and the regularities in [17,18], for more see [13]. Felmer et al. [8] provided the Maximum Principle for small domain to equations involving the fractional Laplacian and then obtained the radial symmetry of positive classical solutions for fractional elliptic equations

$$\begin{cases} (-\Delta)^{\alpha} u = f(u) & \text{in} \quad B_1, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus B_1, \end{cases} \tag{1.2}$$

using the method of moving planes as in [3,10]. The method of moving planes is applied to deal with the overdetermined fractional problems, see [7,25].

The elliptic equations with mixed integro-differential operators $(-\Delta)_x^{\alpha_1} + (-\Delta)_y^{\alpha_2}$, which is the fractional Laplacian of exponent $\alpha_1 \in (0, 1)$ with respect to x and the fractional Laplacian of exponent $\alpha_2 \in (0, 1)$ with respect to y, modeling diffusion sensible to the direction, are associated to Brownian and Levy–Itô processes. Barles, Chasseigne, Ciomaga and Imbert in [1,2] and Ciomaga in [5] considered the existence and the regularity of solutions of equations involving mixed integro-differential operators. Later on, Felmer and Wang studied the decay and the symmetry properties of positive solutions to the mixed integro-differential equations in the whole space. In the present paper, we are interested in the symmetry results of positive solutions for mixed integro-differential equation in a bounded domain, that is,

$$\begin{cases} (-\Delta)_x^{\alpha_1} u + (-\Delta)_y^{\alpha_2} u = f(u) & \text{in} \quad B_1^N(0) \times B_1^M(0), \\ u = 0 \quad \text{in} \quad (\mathbb{R}^N \times \mathbb{R}^M) \setminus (B_1^N(0) \times B_1^M(0)), \end{cases} \tag{1.3}$$

where N, $M \geq 1$, $x \in B_1^N(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, $y \in B_1^M(0) = \{y \in \mathbb{R}^M : |y| < 1\}$, the operators $(-\Delta)_x^{\alpha_1}$ and $(-\Delta)_y^{\alpha_2}$ are given by

$$(-\Delta)_x^{\alpha_1} u(x, y) = P.V. \int_{\mathbb{R}^N} \frac{u(x, y) - u(z, y)}{|x - z|^{N+2\alpha_1}} dz \tag{1.4}$$

and

$$(-\Delta)_y^{\alpha_2} u(x, y) = P.V. \int_{\mathbb{R}^M} \frac{u(x, y) - u(x, \tilde{z})}{|y - \tilde{z}|^{M+2\alpha_2}} d\tilde{z}, \tag{1.5}$$

for all $(x, y) \in B_1^N(0) \times B_1^M(0)$. Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows.

Before stating our main result we make precise the notion of solution that we use in this paper. We say that a continuous function $u : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is a classical solution of equation (1.3) if $(-\Delta)_x^{\alpha_1} u$ and $(-\Delta)_y^{\alpha_2} u$ are defined at any point of $B_1^N(0) \times B_1^M(0)$, according to the definitions given in (1.4) and (1.5), and if u satisfies the equation and the external condition in a pointwise sense.

Now we are ready for our main theorem on symmetry results of positive solutions of equation (1.3). It states as follows: