J. Math. Anal. Appl. 438 (2016) 920–945

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Strongly regular sequences and proximate orders

Javier Jiménez-Garrido, Javier Sanz *

Departamento de Álgebra, Análisis Matemático, Geometría y Topología, Instituto de Investigación en Matemáticas de la Universidad de Valladolid, IMUVA, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain

A R T I C L E I N F O

Article history:
Received 21 October 2015
Available online 9 February 2016
Submitted by U. Stadtmueller

Keywords:
Strongly regular sequences
Regular variation
Proximate orders
Ultraholomorphic classes
Watson's Lemma

A B S T R A C T

Summability methods for ultraholomorphic classes in sectors, defined in terms of a strongly regular sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$, have been put forward by A. Lastra, S. Malek and the second author [28]. We study several open questions related to the existence of kernels of summability constructed by means of analytic proximate orders. In particular, we give a simple condition that allows us to associate a proximate order with a strongly regular sequence. Under this assumption, and through the characterization of strongly regular sequences in terms of so-called regular variation, we show that the growth index $\gamma(\mathbb{M})$ defined by V. Thilliez [54] and the order of quasianalyticity $\omega(\mathbb{M})$ introduced by the second author [50] are the same.

1. Introduction

The study of the existence and meaning of formal power series solutions to differential equations has a long history, going back at least to the works of L. Euler in the 18th century. Although these solutions are frequently divergent, under fairly general conditions the rate of growth of their coefficients is not arbitrary. Indeed, a remarkable result of E. Maillet [32] in 1903 states that any solution $\hat{f} = \sum_{p\geq 0} a_p z^p$ for an analytic differential equation will be of some Gevrey order, that is, there exist $C, A, k > 0$ such that $|a_p| \leq CA^p(p!)^{1/k}$ for every $p \geq 0$.

These series turn out to be Gevrey asymptotic representations of actual solutions defined in suitable domains, and there is the possibility of reconstructing such analytic solutions from the formal ones by a process known as multisummability (in a sense, an iteration of a finite number of elementary k-summability procedures), developed in the 1980's by J.-P. Ramis, J. Écalle, W. Balser et al. This technique has been proven to apply successfully to a plethora of situations concerning the study of formal power series solutions at a singular point for linear and nonlinear (systems of) meromorphic ordinary differential equations

* Corresponding author.
E-mail addresses: jjjimenez@am.uva.es (J. Jiménez-Garrido), jsanzg@am.uva.es (J. Sanz).

http://dx.doi.org/10.1016/j.jmaa.2016.02.010

in the complex domain (see, to cite but a few, the works [1,6,11,38,49]), for partial differential equations (for example, [2,3,16,34,35,44,53]), as well as for singular perturbation problems (see [4,13,27], among others).

However, it is known that non-multisummable (in the previous sense) formal power series solutions may appear for different kinds of equations. G.K. Immink in [18,19] has considered difference equations with formal power series solutions whose coefficients grow at a precise, intermediate rate between those of Gevrey classes, called 1^+ level. She obtained reconstruction results for actual solutions by the consideration of specific kernels and integral transforms, very well suited for the problem studied. Recently, S. Malek [36] has studied some singularly perturbed small step size difference-differential nonlinear equations whose formal solutions with respect to the perturbation parameter can be decomposed as sums of two formal series, one with Gevrey order 1, the other of 1^+ level, a phenomenon already observed for difference equations [12]. In a different context, V. Thilliez [55] has proven some stability results for algebraic equations whose coefficients belong to a general ultraholomorphic class defined by means of a so-called strongly regular sequence (comprising, but not limiting to, Gevrey classes), stating that the solutions will remain in the corresponding class. All these examples made it interesting for us to provide the tools for a general, common treatment of summability in ultraholomorphic classes in sectors, extending the powerful theory of k-summability. The task, achieved by A. Lastra, S. Malek and the second author [28], consisted in the construction of pairs of kernel functions with suitable asymptotic and growth properties, in terms of which to define formal and analytic Laplace- and Borel-like transforms which allow one to construct the sum of a summable formal power series in a direction. The main inspiration came from the theory of moment summability methods developed by W. Balser in [1, Section 5.5], which had already found its application to the analysis of formal power series solutions of different classes of partial differential equations (for example, by S. Malek [33] and by S. Michalik [40]), and also for so-called moment-partial differential equations, introduced by W. Balser and Y. Yoshino [5] and subsequently studied by S. Michalik [39,41,42]. Our technique has been applied in [28] to the study of the summability properties of some formal solutions for moment partial differential equations, generalizing the work in [41], and in [29] to the asymptotic study of the solutions of a class of singularly perturbed partial differential equations in whose coefficients there appear sums of formal power series in this generalized sense. However, some questions remained unsolved in the construction of such generalized summability methods, and the present paper aims at providing some answers, as we proceed to describe.

The Carleman ultraholomorphic classes $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ we consider are those consisting of holomorphic functions f admitting an asymptotic expansion $\hat{f} = \sum_{p\ge 0} a_p z^p/p!$ in a sectorial region G with remainders suitably bounded in terms of a sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive real numbers (we write $f \sim_{\mathbb{M}} \hat{f}$ and $(a_p)_{p\in\mathbb{N}_0} \in \Lambda_{\mathbb{M}}$). The map sending f to $(a_p)_{p\in\mathbb{N}_0}$ is the asymptotic Borel map $\tilde{\mathcal{B}}$. See Subsection 2.3 for the precise definitions of all these classes and concepts. In order to obtain good properties for these classes, the sequence $\mathbb{M}$ is usually subject to some standard conditions; in particular, we will mainly consider strongly regular sequences as defined by V. Thilliez [54], see Subsection 2.2. The best known example is that of Gevrey classes, appearing when the sequence is chosen to be $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, $\alpha > 0$, and for which we use the notations $\tilde{\mathcal{A}}_\alpha(G)$, Λ_α, $f \sim_\alpha \hat{f}$ and so on, for simplicity. Let us denote by G_γ a sectorial region bisected by the direction $d = 0$ and with opening $\pi\gamma$. It is well known that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_\alpha(G_\gamma) \to \Lambda_\alpha$ is surjective if, and only if, $\gamma \le \alpha$ (Borel–Ritt–Gevrey theorem, see [47,48]), while it is injective if, and only if, $\gamma > \alpha$ (Watson's lemma, see for example [1, Prop. 11]). The second author introduced in [50] a constant $\omega(\mathbb{M}) \in (0, \infty)$, measuring the rate of growth of any strongly regular sequence $\mathbb{M}$, in terms of which Watson's Lemma and Borel–Ritt–Gevrey theorem can be generalized in the framework of Carleman ultraholomorphic classes, as long as the associated function $d_{\mathbb{M}}(t) = \log(M(t))/\log t$, where

$$M(t) := \sup_{p\in\mathbb{N}_0} \log\big(\frac{t^p}{M_p}\big), \quad t > 0,$$