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On Sobolev instability of the interior problem of tomography

M. Bertola [a,b,1], A. Katsevich [c,*,2], A. Tovbis [c,2]

[a] *Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. centre ville, Montréal, Québec, H3C 3J7, Canada*
[b] *Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montréal, Québec, H3G 1M8, Canada*
[c] *University of Central Florida, Department of Mathematics, 4000 Central Florida Blvd., P.O. Box 161364, Orlando, FL 32816-1364, United States*

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ABSTRACT

As is known, solving the interior problem with prior data specified on a finite collection of intervals I_i is equivalent to analytic continuation of a function from I_i to an open set $\mathbf{J}$. In the paper we prove that this analytic continuation can be obtained with the help of a simple explicit formula, which involves summation of a series. Our second result is that the operator of analytic continuation is not stable for any pair of Sobolev spaces regardless of how close the set $\mathbf{J}$ is to I_i. Our main tool is the singular value decomposition of the operator $\mathcal{H}_e^{-1}$ that arises when the interior problem is reduced to a problem of inverting the Hilbert transform from incomplete data. The asymptotics of the singular values and singular functions of $\mathcal{H}_e^{-1}$, the latter being valid uniformly on compact subsets of the interior of I_i, was obtained in [5]. Using these asymptotics we can accurately measure the degree of ill-posedness of the analytic continuation as a function of the target interval $\mathbf{J}$. Our last result is the convergence of the asymptotic approximation of the singular functions in the $L^2(I_i)$ sense. We also present a preliminary numerical experiment, which illustrates how to use our results for reducing the instability of the analytic continuation by optimizing the position of the intervals with prior knowledge.

1. Introduction

Suppose one is interested in imaging a small region of interest (ROI) inside an object using tomography. In order to acquire a complete data set that enables stable reconstruction, one needs to send multiple x-rays through the object from many different directions. In particular, the x-rays that do not pass through the

* Corresponding author.
E-mail addresses: bertola@mathstat.concordia.ca (M. Bertola), Alexander.Katsevich@ucf.edu (A. Katsevich), Alexander.Tovbis@ucf.edu (A. Tovbis).
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ROI are required as well. The interior problem of tomography arises when only the x-rays through the ROI are measured. In this case the tomographic data are incomplete, and image reconstruction becomes a challenging problem. In what follows, image reconstruction from x-ray data taylored to an ROI will be called the interior problem, and the corresponding data will be called interior data. Practical importance of the interior problem is clear, since tayloring the x-ray exposure to an ROI results in a reduced x-ray dose to the patient in medical applications of tomography. See [28] for a nice review of the state of the art in interior tomography.

One of the most powerful tools for investigating the interior problem from the theoretical point of view is the Gelfand–Graev formula, which relates the tomographic data of an object with its one-dimensional Hilbert transform along lines [13]. With the help of this formula, the interior problem of tomography can be reduced to the problem of inverting the Hilbert transform from incomplete data, see e.g. [23,7,32,31,29,30,20,6]. A more recent line of research based on a differential operator that commutes with the Finite Hilbert Transform (FHT) is represented by the papers [16,15,18,2,1].

Let f be sufficiently smooth and compactly supported. The (restricted) cone beam transform of f is defined as a collection of integrals of f along lines intersecting a curve Γ:

$$D_f(y,\beta) = \int_0^\infty f(y+t\beta)dt,\ \ y\in\Gamma, \tag{1.1}$$

where β is a unit vector. We assume that Γ is piecewise smooth and does not intersect the support of f. In practice, Γ is called the source trajectory. The data D_f are collected by moving the x-ray source along Γ, irradiating the object with multiple x-ray beams, and measuring the intensities of the beams after they exit the object.

Let $y(s)$ be a parametrization of Γ. We assume that Γ does not self-intersect and is traversed in one direction as s varies over some interval I. Pick any two values $s_1, s_2 \in I$, $s_1 \neq s_2$. Let α be a unit vector along the chord $y(s_1), y(s_2)$. Then one has [13]:

$$\frac{1}{2}\int_{s_1}^{s_2} \frac{1}{|x-y(s)|}\,\frac{\partial}{\partial\lambda} D_f\left(y(\lambda), \frac{x-y(s)}{|x-y(s)|}\right)\Bigg|_{\lambda=s} ds = \int \frac{f(x+t\alpha)}{t}dt, \tag{1.2}$$

where x is located on the chord between $y(s_1)$ and $y(s_2)$. Equation (1.2) implies that knowing the cone beam transform of f one can compute the Hilbert transform of f on the chords of Γ.

Fix any chord $[y(s_1), y(s_2)]$ of Γ, and let L be the line determined by the chord. Source trajectories that are commonly used in practice have the property that for any point in the object support there is a chord of Γ containing that point. In what follows we regard L as the x-axis. Fix some $2g+2$, $g\in\mathbb{N}$, distinct points a_i on L: $a_i < a_{i+1}$, $i = 1, 2, \ldots, 2g+1$. Points a_1 and a_{2g+2} mark the boundaries of the support of f along L. Points a_2 and a_{2g+1} mark the boundaries of the ROI along L. Consider the FHT

$$(\mathcal{H}f)(x) := \frac{1}{\pi}\int_{a_1}^{a_{2g+2}} \frac{f|_L(y)}{y-x}dy,\ \ f|_L \in L^2([a_1, a_{2g+2}]). \tag{1.3}$$

Here $f|_L$ is the restriction of f to L, and $\mathcal{H}f$ is the one-dimensional Hilbert transform of $f|_L$. Throughout the paper the line L is always the same, so with some abuse of notation we write f instead of $f|_L$. In the case of interior tomographic data, the Gelfand–Graev formula allows computation of $\mathcal{H}f$ only on $[a_2, a_{2g+1}]$, but not on all $[a_1, a_{2g+2}]$. Thus the interior problem of tomography is reduced to finding f inside the ROI, i.e. on $[a_2, a_{2g+1}]$, by solving the equation

$$(\mathcal{H}f)(x) = \varphi(x),\ \ x\in[a_2, a_{2g+1}]. \tag{1.4}$$