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Uniform regularity estimates in homogenization theory of elliptic system with lower order terms ☆

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ABSTRACT

In this paper, we extend the uniform regularity estimates obtained by M. Avellaneda and F. Lin in [3,6] to the more general second order elliptic systems in divergence form $\{\mathcal{L}_\varepsilon, \varepsilon > 0\}$, with rapidly oscillating periodic coefficients. We establish not only sharp $W^{1,p}$ estimates, Hölder estimates, Lipschitz estimates and non-tangential maximal function estimates for the Dirichlet problem on a bounded $C^{1,\eta}$ domain, but also a sharp $O(\varepsilon)$ convergence rate in $H_0^1(\Omega)$ by virtue of the Dirichlet correctors. Moreover, we define the Green's matrix associated with $\mathcal{L}_\varepsilon$ and obtain its decay estimates. We remark that the well known compactness methods are not employed here, instead we construct the transformations (1.11) to make full use of the results in [3,6].

1. Introduction and main results

The main purpose of this paper is to study the uniform regularity estimates for second order elliptic systems with lower order terms, arising in homogenization theory. More precisely, we consider

$$\mathcal{L}_\varepsilon = -\mathrm{div}\left[A\left(x/\varepsilon\right)\nabla + V(x/\varepsilon)\right] + B(x/\varepsilon)\nabla + c(x/\varepsilon) + \lambda I,$$

where λ is a constant, and $I = (\delta^{\alpha\beta})$ denotes the identity matrix. In a special case, let $A = I = 1$, $V = B$, $c = 0$, and $\mathcal{W} = \mathrm{div}(V)$, the operator $\mathcal{L}_\varepsilon$ becomes

$$\mathfrak{L}_\varepsilon = -\Delta + \frac{1}{\varepsilon}\mathcal{W}(x/\varepsilon) + \lambda,$$

where $\mathcal{W}$ is the rapidly oscillating potential term (see [7, p. 91]). It is not hard to see that the uniform regularity estimates obtained in this paper are not trivial generalizations of [3,6], and they are new even for $\mathfrak{L}_\varepsilon$.

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Let $1 \le i, j \le d$, $1 \le \alpha, \beta \le m$, where $d \ge 3$ denotes the dimension, and $m \ge 1$ is the number of equations in the system. Suppose that the measurable functions $A = (a_{ij}^{\alpha\beta}) : \mathbb{R}^d \to \mathbb{R}^{m^2 \times d^2}$, $V = (V_i^{\alpha\beta}) : \mathbb{R}^d \to \mathbb{R}^{m^2 \times d}$, $B = (B_i^{\alpha\beta}) : \mathbb{R}^d \to \mathbb{R}^{m^2 \times d}$, $c = (c^{\alpha\beta}) : \mathbb{R}^d \to \mathbb{R}^{m^2}$ satisfy the following conditions:

- the uniform ellipticity condition

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \mu^{-1}|\xi|^2 \quad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{md}, \text{ where } \mu > 0; \tag{1.1}$$

(The summation convention for repeated indices is used throughout.)

- the periodicity condition

$$A(y+z) = A(y),\ V(y+z) = V(y),\ B(y+z) = B(y),\ c(y+z) = c(y) \quad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d; \tag{1.2}$$

- the boundedness condition

$$\max\{\|V\|_{L^\infty(\mathbb{R}^d)}, \|B\|_{L^\infty(\mathbb{R}^d)}, \|c\|_{L^\infty(\mathbb{R}^d)}\} \le \kappa_1, \quad \text{where } \kappa_1 > 0; \tag{1.3}$$

- the regularity condition

$$\max\{\|A\|_{C^{0,\tau}(\mathbb{R}^d)},\ \|V\|_{C^{0,\tau}(\mathbb{R}^d)},\ \|B\|_{C^{0,\tau}(\mathbb{R}^d)}\} \le \kappa_2, \quad \text{where } \tau \in (0,1) \text{ and } \kappa_2 > 0. \tag{1.4}$$

Set $\kappa = \max\{\kappa_1, \kappa_2\}$, and we say $A \in \Lambda(\mu, \tau, \kappa)$ if $A = A(y)$ satisfies conditions (1.1), (1.2) and (1.4). Throughout this paper, we always assume that Ω is a bounded $C^{1,\eta}$ domain with $\eta \in [\tau, 1)$, and $L_\varepsilon = -\mathrm{div}[A(x/\varepsilon)\nabla]$ is the elliptic operator from [3,6], unless otherwise stated.

The main idea of this paper is to find the transformations (1.11) between two solutions corresponding to $\mathcal{L}_\varepsilon$ and L_ε such that the regularity results of L_ε can be applied to $\mathcal{L}_\varepsilon$ directly. Particularly, to handle the boundary Lipschitz estimates, we define the Dirichlet correctors $\Phi_{\varepsilon,k} = (\Phi_{\varepsilon,k}^{\alpha\beta})$, $0 \le k \le d$, associated with $\mathcal{L}_\varepsilon$ as follows:

$$L_\varepsilon(\Phi_{\varepsilon,k}) = \mathrm{div}(V_\varepsilon) \ \text{ in } \Omega, \qquad \Phi_{\varepsilon,k} = I \ \text{ on } \partial\Omega \tag{1.5}$$

for $k = 0$, and

$$L_\varepsilon(\Phi_{\varepsilon,k}^\beta) = 0 \ \text{ in } \Omega, \qquad \Phi_{\varepsilon,k}^\beta = P_k^\beta \ \text{ on } \partial\Omega \tag{1.6}$$

for $1 \le k \le d$, where $V_\varepsilon(x) = V(x/\varepsilon)$, $\Phi_{\varepsilon,k}^\beta = (\Phi_{\varepsilon,k}^{1\beta}, \cdots, \Phi_{\varepsilon,k}^{m\beta}) \in H^1(\Omega; \mathbb{R}^m)$, and $P_k^\beta = x_k(0, \cdots, 1, \cdots, 0)$ with 1 in the βth position. We remark that (1.6) was studied in [3,39], but (1.5) has not yet been developed. Here we show that $\Phi_{\varepsilon,0}$ ought to be of the form in (1.5), and its properties are shown in section 4.

For Neumann boundary conditions, a significant development was made by C.E. Kenig, F. Lin and Z. Shen [32], where they constructed Neumann correctors to verify the Lipschitz estimates for L_ε. Recently, S.N. Armstrong and Z. Shen [2] found a new way to obtain the same results even without Dirichlet correctors or Neumann correctors in the almost periodic setting. We plan to study uniform regularity estimates for $\mathcal{L}_\varepsilon$ with Neumann boundary conditions in a forthcoming paper.

The main results are as follows.

Theorem 1.1 *($W^{1,p}$ estimates). Suppose that $A \in \mathrm{VMO}(\mathbb{R}^d)$ satisfies (1.1), (1.2), and other coefficients of $\mathcal{L}_\varepsilon$ satisfy (1.3). Let $1 < p < \infty$, $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{md})$, $F \in L^q(\Omega; \mathbb{R}^m)$ and $g \in B^{1-\frac{1}{p},p}(\partial\Omega; \mathbb{R}^m)$, where $q = \frac{pd}{p+d}$ if $p > \frac{d}{d-1}$, and $q > 1$ if $1 < p \le \frac{d}{d-1}$. Then the Dirichlet problem*