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Extreme points of the Harsanyi set and the Weber set

Genjiu Xu [a,*], Theo S.H. Driessen [b], Jun Su [c], Hao Sun [a]

[a] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

[b] Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

[c] School of Science, Xi'an University of Science and Technology, Xi'an, Shaanxi 710054, PR China

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ABSTRACT

In this paper, we present firstly a matrix approach, by Moebius transformation, to axiomatize the Harsanyi payoff vectors in the traditional worth system instead of the dividend system. Then by this approach, the Weber set is also characterized as the set of specialized Harsanyi payoff vectors. The study of marginal contribution vectors, the extreme points of the Weber set is pivotal to characterize the Weber set. Recall that an extreme point of a linear system can be recognized by its carriers. A linear system associated to the Weber set is constructed and a second approach to investigate their extreme points is accessed by the concept of carrier. We apply the same technique to study the extreme points of the Harsanyi set. Together with the core-type structure of the Harsanyi set, we present a recursive algorithm for computing the extreme points of the Harsanyi set for any game.

1. Introduction

In economic situations, players may cooperate to obtain more profits with assuming that they are rational. It is a common and important issue to distribute the surplus of cooperation among the players. Cooperative game theory provides general mathematical techniques for analyzing such cooperation and distribution issues. The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings (worth) among the players in the game.

A solution is a mapping which assigns to every game a set of payoff allocations over the players in the game. The Core and the Weber set are two well-known set-valued solutions. The core (Gillies [9]) of a game is the set of feasible allocations that cannot be improved upon by a subset (a coalition) of the players. The Weber set (Weber [26]) is the solution mapping that assign to each game the set of convex combinations

* Corresponding author. Fax: +86 29 88431657.
E-mail addresses: xugenjiu@nwpu.edu.cn (G. Xu), t.s.h.driessen@utwente.nl (T.S.H. Driessen), junsu99@126.com (J. Su), hsun@nwpu.edu.cn (H. Sun).

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of the marginal contribution vectors, consisting of all random order values. These two set-valued solutions coincide on convex games.

Different from most of solutions considering to allocate the worth of coalitions in cooperative game theory, the Harsanyi set (Vasil'ev [21,23]), consisting of all sharing values, is a solution mapping to share the dividends of coalitions. The dividend is first discussed in Harsanyi [13,14], as a notion that captures the value of a coalition which is solely acquired by the cooperation of all players within the coalition. This solution is considered first in Hammer et al. [12] (as the Selectope) and, independently, in Vasil'ev [21,23]. Meanwhile, it is proved that the Harsanyi set encloses the core of the game [12,21], and the Harsanyi set has a core-type structure (Vasil'ev [23] and Derks et al. [3,5]). Many researches have addressed on the relations between sharing values and random order values, as well as the Harsanyi set and the Weber set (Vasil'ev and van der Laan [25], Derks and Peters [6], Derks et al. [4]). In particular, Derks et al. [3] have shown that the Weber set is a subset of the Harsanyi set and also provide conditions on a game for the Harsanyi set to coincide with the Weber set.

From the algebraic point of view, the dividend and the worth are both representations for a game with respect to different basis systems on the game space. There is a linear operator that transforms one of them to the other, i.e., Moebius transformation (Grabisch [10]). We are motivated to investigate the relations between solutions defined on dividend and worth by means of the Moebius transformation. Moreover, both a sharing value and a random order value are linear operators on game space. Matrix approach has been revealed as a natural and powerful technique for studying linear operators in cooperative game theory (Xu el al. [27] and Hamiache [11]). Stimulated by this, we aim to take use of the matrix approach, to provide new and intuitive proofs for the results mentioned above, and we reposition several others as well as derive some new results within a more general framework.

This paper mainly provides new and intuitive proofs of some known results using the matrix approach to cooperative games. By studying the properties of different types of matrix, we illustrate how a Harsanyi payoff vector to allocate the worth or to share the complementary dividends instead of the dividends. On the other hand, the Weber set, consisting of all random order values, is also characterized how to share the dividends. The matrix approach yields insight into these solutions, and it will be helpful to get more properties of solutions.

The paper is organized as follows. Section 2 states the game theoretic notions being further discussed. In Section 3 (complementary) Moebius transformation matrices are introduced and analyzed, for illustrating the relations between the worth and the (complementary) dividend with respect to different bases for the game space. In Section 4 we develop a matrix approach, by Moebius transformation, to axiomatize the Harsanyi payoff vectors in the traditional worth system instead of the dividend system. And in Section 5 a recursive algorithm for computing the extreme points of the Harsanyi set is proposed in terms of its core-type structure. In Section 6, with the analysis of marginal vectors, the extreme points of the Weber set, we provide a matrix approach to characterize the Weber set as the set of specialized Harsanyi payoff vectors. We also recognize these extreme points by the notion of carrier of an introduced linear system in Section 7. Section 8 concludes the paper.

2. Preliminaries

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. An element of N and a subset S of N (i.e., $S \in 2^N$) are called a *player* and a *coalition* respectively. The associated real number $v(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by s. We denote by $\mathcal{G}^N$ the set of all these TU-games with player set N and by $\Omega = 2^N \setminus \{\emptyset\}$ the set of all nonempty coalitions.