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Spectral deformations and exponential decay of eigenfunctions for the Neumann Laplacian on manifolds with quasicylindrical ends

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ABSTRACT

We study spectral properties of the Neumann Laplacian on manifolds with quasicylindrical ends. In particular, we prove exponential decay of the non-threshold eigenfunctions and show that the eigenvalues can accumulate only at thresholds of the absolutely continuous spectrum and only from below. The non-threshold eigenvalues are also discrete eigenvalues of a non-selfadjoint operator.

1. Introduction and statement of results

We consider manifolds with dilation analytic quasicylindrical ends and non-compact smooth boundary. These manifolds correspond to a class of long-range geometric perturbations of manifolds with cylindrical ends. For mild geometric perturbations spectral properties and eigenfunction asymptotics of elliptic problems are essentially well known, e.g. [2,4,6,8,14–17]. The perturbations we consider here are so strong that eigenfunctions do not admit asymptotic expansions at the ends and can only be represented as a sum of norm convergent operator series plus a remainder, e.g. [15]. Thus the asymptotic theory per se does not give any information on decay of eigenfunctions. Besides, examples demonstrate that eigenvalues can accumulate at finite values of the spectral parameter.

We study the spectrum of the Neumann Laplacian and prove exponential decay of eigenfunctions by the method of spectral deformations originating from the theory of resonances for N-body quantum scattering, see e.g. [3,7,19]. Spectral deformations (also known as *complex scaling* and *analytic dilations*) are also traditionally used in numerical analysis [3]. In 1994 a modification of this approach was reinvented by

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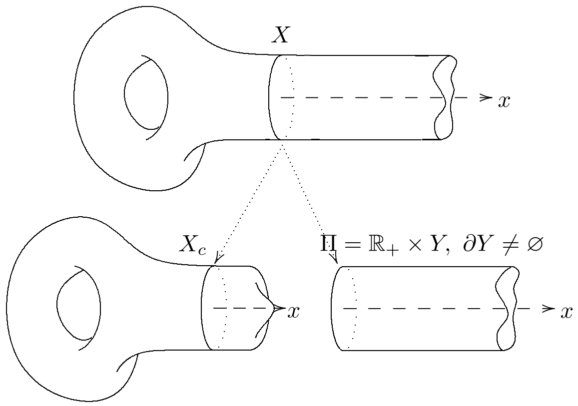

Fig. 1. Representation $X = X_c \cup \Pi$.

Bérenger [1] under the name of Perfectly Matched Layer (PML) method and became very popular since then. In our study of the PML method for the Neumann Laplacian [10] we already developed spectral deformations on manifolds with quasicylindrical ends to some extent: we introduced proper geometric deformations in quasicylindrical ends (or, equivalently, constructed infinite PMLs in the quasicylindrical ends) and studied corresponding deformations of the essential spectrum. As the main results of this work we prove that 1) The eigenvalues are of finite multiplicity and can accumulate only at thresholds of the continuous spectrum and only from below (Assertion 8 of Theorem 1 below); 2) Any non-threshold eigenfunction decays with an exponential rate that depends on the distance from the corresponding eigenvalue to the next threshold of continuous spectrum (Assertion 9 of Theorem 1 below).

Let Y be a compact n-dimensional manifold with smooth boundary ∂Y. Denote by Π the semi-cylinder defined as the Cartesian product $\mathbb{R}_+ \times Y$, where $\mathbb{R}_+$ is the positive semi-axis. Consider an oriented $(n+1)$-dimensional manifold X representable in the form $X = X_c \cup \Pi$, where X_c is a compact manifold with smooth boundary, see Fig. 1. We assume that the boundary ∂X of X is smooth.

Let g be a Riemannian metric on X. We identify the cotangent bundle $\mathrm{T}^*\Pi$ with the tensor product $\mathrm{T}^*\mathbb{R}_+ \otimes \mathrm{T}^*Y$ via the natural isomorphism induced by the product structure on Π. Then

$$\mathsf{g}{\restriction}_\Pi = \mathfrak{g}_0 dx^2 + 2\mathfrak{g}_1 dx + \mathfrak{g}_2, \quad \mathfrak{g}_k(x) \in C^\infty \mathrm{T}^*Y^{\otimes k},\ x \in \mathbb{R}_+. \tag{1}$$

Let $\mathbb{C}\mathrm{T}^*Y^{\otimes k}$ stand for the tensor power of the complexified cotangent bundle $\mathbb{C}\mathrm{T}^*Y$ with fibers $\mathbb{C}\mathrm{T}^*_yY = \mathrm{T}^*_yY \otimes \mathbb{C}$. In what follows C^m stands for sections of complexified bundles. We equip the space $C^1\mathrm{T}^*Y^{\otimes k}$ with the norm

$$\|\cdot\|_\mathfrak{e} = \max_{y\in Y}\big(|\cdot|_\mathfrak{e}(y) + |\nabla \cdot|_\mathfrak{e}(y)\big), \tag{2}$$

where $\mathfrak{e}$ is a Riemannian metric on Y, $|\cdot|_\mathfrak{e}(y)$ is the norm in $\mathbb{C}\mathrm{T}^*_yY^{\otimes k}$, and $\nabla : C^1\mathrm{T}^*Y^{\otimes k} \to C^0\mathrm{T}^*Y^{\otimes k+1}$ is the Levi-Civita connection on $(Y, \mathfrak{e})$. By taking a different metric $\mathfrak{e}$ on Y one gets an equivalent norm $\|\cdot\|_\mathfrak{e}$.

Definition 1. We say that (X, g) is a manifold with dilation analytic quasicylindrical end $(\Pi, \mathsf{g}{\restriction}_\Pi)$, if the following conditions hold:

i. The coefficients $\mathbb{R}_+ \ni x \mapsto \mathfrak{g}_k(x) \in C^\infty\mathrm{T}^*Y^{\otimes k}$ in (1) extend by analyticity from the semi-axis $\mathbb{R}_+$ to the sector $\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}$.
ii. The values $\|\mathfrak{g}_0(z) - 1\|_\mathfrak{e}$, $\|\mathfrak{g}_1(z)\|_\mathfrak{e}$, and $\|\mathfrak{g}_2(z) - \mathfrak{h}\|_\mathfrak{e}$ converge to zero uniformly in $z \in \mathbb{S}_\alpha$ as $|z| \to \infty$, where $\mathfrak{h}$ is a metric on Y.