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Comparison meaningful operators and ordinal invariant preferences

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A B S T R A C T

The existence of a continuous and order-preserving real-valued function, for the class of continuous and ordinal invariant total preorders, defined on the Banach space of all bounded real-valued functions, which are in turn defined on a given set Ω, is characterized. Whenever the total preorder is nontrivial, the type of representation obtained leads to a functional equation that is closely related to the concept of comparison meaningfulness, and is studied in detail in this setting. In particular, when restricted to the space of bounded and measurable real-valued functions, with respect to some algebra of subsets of Ω, we prove that, if the total preorder is also weakly Paretian, then it can be represented as a Choquet integral with respect to a $\{0, 1\}$-valued capacity. Some interdisciplinary applications to measurement theory and social choice are also considered.

1. Introduction

A meaningful statement is an expression that comes from measurement theory and follows the so-called Luce's principle of dimensional analysis (also termed as the principle of theory construction). According to this, a statement (usually involving a formula or an equation) is meaningful if admissible transformations of the input variables (the scale by which input variables are measured) lead to admissible transformations of the output variables (the scale by which output variables are measured). Whenever this happens, then it is said that *comparison meaningfulness* is satisfied. For an account of measurement theory, see [13] and [15].

Let Ω stand for a given nonempty set. Let $\mathbb{R}^\Omega$ denote the set of all real-valued functions defined on Ω, and let B_Ω be the subset of $\mathbb{R}^\Omega$, which consists of all bounded functions. Let $\precsim$ be a total preorder (also known as a preference) on B_Ω. In simple terms, $\precsim$ is *ordinal invariant* if the ranking between any two functions of B_Ω does not change whenever the values of these two functions are measured on the same ordinal scale. In

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formula, $\precsim$ is ordinal invariant provided that $f \precsim g \Rightarrow \phi \circ f \precsim \phi \circ g$, for every $f, g \in B_\Omega$, $\phi \in \Phi$, where Φ stands for the set that consists of all strictly increasing real-valued functions of a single variable.

A real-valued function $T : B_\Omega \to \mathbb{R}$ is said to be an *operator* (or a *functional*). An operator T is said to be *ordinal covariant* (respectively, *negatively ordinal covariant*) if $T(\phi \circ f) = \phi(T(f))$ (respectively, if $T(\phi \circ f) = -\phi(-T(f)))$ holds true for every $f \in B_\Omega$, $\phi \in \Phi$. Note that these two concepts define corresponding functional equations to be satisfied by T. As will be seen later, a key class of ordinal invariant total preorders; viz., the continuous and representable ones are naturally linked to these types of operators.

Ordinal covariant operators have some interesting features. On the one hand, they satisfy a property, mentioned above, that plays an important role in the theory of aggregation functions; namely, comparison meaningfulness. On the other hand, in the finite dimensional case, ordinal covariant operators are closely related to lattice polynomial functions. Lattice polynomial functions were introduced by Birkhoff [2]. Recently, they have been completely characterized (see [17,20]). Roughly speaking, a lattice polynomial function on $\mathbb{R}^n$ is a Boolean max–min map, i.e., a real-valued function of n variables that is obtained by computing maxima and minima according to a fixed collection of subsets of variables. In [18], Marichal and Mesiar provide an excellent survey regarding meaningful aggregation functions mapping ordinal scales into an ordinal scale. These authors study three classes of aggregation functions defined on particular subsets of the Euclidean space, including certain interpretations of the main functional equations involving these classes in the setting of aggregation on finite chains.

Corresponding extensions of these results in the context of B_Ω^Σ have been given by Ovchinnikov and Dukhovny (see [21] and [22]). Here, Σ is a nonempty algebra of subsets of Ω, and $B_\Omega^\Sigma \subseteq B_\Omega$ is the space of all bounded measurable real-valued functions on Ω. In particular, in the latter articles, it is proven that a continuous functional on B_Ω^Σ is invariant (under transformations from the automorphism group of the set of all real numbers) if and only if it can be represented as a Choquet or Sugeno (fuzzy) integral with respect to a $\{0, 1\}$-valued capacity.[1] Similar results appear in [8], where the continuity assumption is replaced with a monotonicity condition. In addition, the connection between this type of operator and the so-called probabilistic quantiles, which are of significant prevalence in statistics, is also studied.

The main purpose of the current paper, quite different from the starting-point of the articles just mentioned, is to offer an account of the continuous representation problem for the class of continuous and ordinal invariant preferences defined on B_Ω (B_Ω is equipped with the supremum norm topology). Thus, we work in a framework more general than those mentioned above, including the finite-dimensional case. In addition, we show the significance of our characterization results in social sciences by introducing certain applications in measurement theory and social choice theory.

Here is a brief outline of the contents of the paper. In Section 2 we introduce the basics regarding orders and operators on B_Ω. In Section 3 our main theorem is shown: Continuous and ordinal invariant total preorders defined on B_Ω are identified with those that admit a continuous and ordinal covariant utility (order-preserving) function, or a continuous and negatively ordinal covariant utility function, or are trivial. When restricted to B_Ω^Σ, by taking advantage of the results in [21] and [22], we can offer the following finer statement: If the total preorder is also weakly Paretian, then it can be represented as a Choquet integral with respect to a $\{0, 1\}$-valued capacity.

In Section 4, and as a consequence of our main theorem, we present some results in the field of measurement theory. In particular, we generalize a result by Marichal and Mathonet [17] (see also [16]) about the characterization of the class of continuous and *comparison meaningful operators*. Moreover, certain monotonicity properties of these operators are also shown. The corresponding results for continuous and ordinal invariant total preorders defined on B_Ω are also presented.

As a second application, we also offer a characterization of certain social rules, called *social evaluation functionals*, in the setting of utility theory in social choice. In particular, we study a slight deviation of the

[1] We are grateful to an anonymous referee for bringing our attention to Refs. [18,21,22].