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A nonlinear weighted least-squares finite element method for the Carreau–Yasuda non-Newtonian model

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ABSTRACT

We study a nonlinear weighted least-squares finite element method for the Navier–Stokes equations governing non-Newtonian fluid flows by using the Carreau–Yasuda model. The Carreau–Yasuda model is used to describe the shear-thinning behavior of blood. We prove that the least-squares approximation converges to linearized solutions of the non-Newtonian model at the optimal rate. By using continuous piecewise linear finite element spaces for all variables and by appropriately adjusting the nonlinear weighting function, we obtain optimal L^2-norm error convergence rates in all variables. Numerical results are given for a Carreau fluid in the 4-to-1 contraction problem, revealing the shear-thinning behavior. The physical parameter effects are also investigated.

1. Introduction

The objective of this study is to analyze a nonlinear weighted least-squares finite element method for the Carreau–Yasuda non-Newtonian model based on the Navier–Stokes equations. The Carreau–Yasuda model is a popular non-Newtonian model for describing the shear-thinning behavior of blood in hemodynamic simulations [5,16].

Let Ω be an open, connected, and bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 with boundary Γ. The steady-state, incompressible Navier–Stokes equation with the velocity boundary condition can be posed as follows:

$$\begin{aligned} \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\boldsymbol{\tau} + \nabla p &= \hat{\mathbf{f}} \quad \text{in } \Omega, \\ \boldsymbol{\tau} - \frac{2\eta\left(\dot{\gamma}(\mathbf{u})\right)\mathbf{D}(\mathbf{u})}{\eta_0 Re} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

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where $\mathbf{D}(\mathbf{u}) = 0.5(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the standard strain rate tensor. $Re \geq 1$ is the Reynolds number, $Re \equiv LU\rho/\eta_0$, in which η_0 is the zero-shear-rate viscosity, L and U are characteristic length and velocity, respectively, and ρ is the density. $\hat{\mathbf{f}}$ is the body force vector, the unknowns $\mathbf{u}$ and $\boldsymbol{\tau}$ are the velocity and the extra-stress tensor, respectively, and p is the scalar pressure. We assume that the pressure p satisfies a zero mean constraint:

$$\int_\Omega p dx = 0,$$

in order to assure the uniqueness of pressure [2]. As for the system (1), it is illustrated in [4] that the system is suitable for incompressible non-Newtonian flows when a direct approximation of the extra stress tensor is desired.

Let $\dot{\gamma}(\mathbf{u}) = \sqrt{2\left(\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u})\right)}$ be the shear rate with the double-dot product between two second-order tensors $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ defined as

$$\boldsymbol{\tau} : \boldsymbol{\sigma} = \sum_{i,j} \boldsymbol{\tau}_{ij}\boldsymbol{\sigma}_{ji}.$$

We implement the non-Newtonian fluid equation known as the Carreau–Yasuda model [5], i.e.

$$\eta(\dot{\gamma}(\mathbf{u})) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\lambda_c\dot{\gamma}(\mathbf{u}))^a]^{\frac{n-1}{a}}, \tag{2}$$

where a, n, and λ_c are determined constant parameters. $a > 0$ is the dimensionless parameter, λ_c is the Carreau time constant, and the parameter n is the power law exponent. In the case of $n = 1$, the model reduces to the linear Newtonian model, i.e. the Navier–Stokes equations. For a shear-thinning fluid, n is less than one, the viscosity decreases by increasing shear rate. At high shear rates, the viscosity of the fluid is η_∞, whereas at low shear rates, the viscosity is η_0. Sample values of the parameters in the Carreau–Yasuda model are given in [1]. They indicate that many concentrated polymer solutions and melts can be obtained for $a = 2$ and $\eta_\infty = 0$. Equation (2), with $a = 2$, is usually referred to as the Carreau equation, and the parameter a is added later by Yasuda; see [1].

Numerous developments using least-squares finite element methods for non-Newtonian fluid flow problems have been made in recent years [4,6,8–12]. Least-squares finite element methods have been reported to offer several theoretical and computational advantages over the Galerkin method for various boundary value problems [2]. Discretization generates an algebraic system that is always symmetric and positive definite, and a single approximating space for all variables can be used for programming least-squares finite element methods [14]. The least-squares functional of the velocity–pressure–stress formulation has the advantage that stress tensor components are computed directly [13]. Hence, the method is suitable for cases in which a direct approximation of the extra stress tensor is necessary (e.g., non-Newtonian fluid flows).

In [4], Bose and Carey present a least-squares method using p-type finite elements and a mesh redistribution for non-Newtonian flows, and indicate the importance of scaling in the original differential equations for the least-squares minimization process. In [14], Lee and Chen propose a nonlinear weighted least-squares (NL-WDLS) method that allows for the use of simple combinations of interpolations, including equal-order linear elements for Stokes equations. They indicate the choice of weights used to balance the residual contributions, and their results show some improvement over the case with no weightings. On the basis of their ideas, NL-WDLS methods based on the velocity–stress–pressure formulation of Stokes equations have been applied to generalized Newtonian and viscoelastic fluid flows in numerical experiments [8,11]. The results indicate that when linear approximations in all variables are employed, the least-squares solutions exhibit numerical convergence rates of $O(h^2)$ in the L^2-norm for all dependent variables (or nearly so for the viscoelastic case). In [12], an adaptively refined least-squares (AR-LS) approach with an inertial term