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Existence of a global solution to a scalar conservation law with a source term for large data

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ABSTRACT

We are concerned with a scalar conservation law with a source term. This equation is proposed to describe the qualitative behavior of waves for a general system in resonance with the source term by T.P. Liu. The goal in the present paper is to provide a condition that the Cauchy problem has a global entropy solution for large data. The key point is to obtain the bounded estimate of solutions. To deduce this, we introduce some functions of x as the lower and upper bounds. Therefore, our bounded estimate depends on the space variable. Moreover, we use the vanishing viscosity method to construct approximate solutions and derive the convergence by the compensated compactness.

1. Introduction

We study the following scalar conservation law with a source term

$$u_t + f(u)_x = a(x)g(u), \quad u(x,t) \in \mathbf{R},\ x \in \mathbf{R},\ t > 0, \tag{1.1}$$

and initial data

$$u(x,0) = u_0(x), \quad u_0 \in L^\infty(\mathbf{R}), \tag{1.2}$$

where a is a given function. This equation is proposed to describe the qualitative behavior of waves for a general system in resonance with the source term by T.P. Liu (see [3]).

We assume that

(A1) $a \in \mathcal{B}^1(\mathbf{R})$ and $\int_{-\infty}^{\infty} |a(x)|^p dx < \infty$ for some $0 < p \leqq 1$. There exists a positive function $b \in \mathcal{B}^1(\mathbf{R}) \cap L^1(\mathbf{R})$ satisfying $|a(x)|^p \leq b(x)$.

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(A2) $g \in C^1(\mathbf{R})$.
(A3) $f \in C^2(\mathbf{R})$, $f'(0) = 0$, $f''(u) > 0$.

Here $C^k(\mathbf{R})$ denotes the space of the functions whose derivatives up to k-th order are continuous; $\mathcal{B}^k(\mathbf{R})$ denotes the space of the functions whose derivatives up to k-th order are continuous and bounded over $\mathbf{R}$, equipped with the norm $|f|_k = \sum_{i=0}^{k} \sup_{x\in\mathbf{R}} \left| \frac{d^i f}{dx^i}(x) \right|$.

As mentioned in [3], we are interested in the nonlinear resonance, that is, the characteristic speed $f'(u)$ of (1.1) is around zero. Since

$$f'(u) > 0 \text{ for } u > 0 \text{ and } f'(u) < 0 \text{ for } u < 0, \tag{1.3}$$

we consider a solution including zero.

We review the relative results for (1.1). The pioneer work in this direction is [3]. In [3], T.P. Liu investigated a global solution with a bounded total variation by the Glimm scheme. On the other hand, in [2], the L^∞ and L^2 solutions were constructed by the compensated compactness. In this result, $0 \leqq a(x)$ is further needed. Moreover, $g'(u)$ has a constant sign in [2] and [3].

In this paper, we do not need these restrictions of $a(x)$ and $g(u)$. In [7], under such a condition, the global existence of a solution was proved for small data. The goal in the present paper is to provide a condition that the Cauchy problem (1.1)–(1.2) has a global solution for large data.

The key point of this paper is to obtain the bounded estimate, Theorem 2.2. To do this, we further assume the following condition.

(A4) For any positive value M_0, there exist positive constants k and M such that

$$M \geqq M_0 + \frac{\delta}{3}, \tag{1.4}$$

$$M > \delta,$$

$$kf'(u) + \|a\|_\infty^{1-p}|g(u)| < 0 \text{ for } u \in (-M-\delta, -M+\delta) \tag{1.5}$$

and

$$kf'(u) - \|a\|_\infty^{1-p}|g(u)| > 0 \text{ for } u \in (M-\delta, M+\delta), \tag{1.6}$$

where $\delta = 3k\|b\|_1$. In addition, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ represent the norms on $L^\infty(\mathbf{R})$ and $L^1(\mathbf{R})$ respectively.

(A4) is our desired condition. We give some remarks for this condition.

Remark 1.1.

1. If $a(x) \equiv 0$, from [1] and [3], the Cauchy problem (1.1)–(1.2) has a global solution for large data. On the other hand, we notice that (1.5)–(1.6) apparently holds in this case.
2. We next consider the relation between (1.5)–(1.6) and the order of $g(u)/f'(u)$ as $|u| \to \infty$.
   (a) If $\dfrac{f'(u)}{g(u)} = \dfrac{C(1+o(1))}{|u|^q}$ for some $q < 1$, (1.5) and (1.6) hold;
   (b) If $\dfrac{f'(u)}{g(u)} = \dfrac{C(1+o(1))}{|u|}$ and $\|a\|_\infty^{1-p}\|b\|_1$ is small enough, (1.5) and (1.6) hold,
   where C is a constant and $o(1) \to 0$ as $|u| \to \infty$.