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Sets of uniqueness for uniform limits of polynomials in several complex variables

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ABSTRACT

We investigate the sets of uniform limits $A(\overline{B}_n)$, $A(\overline{D}^I)$ of polynomials on the closed unit ball $\overline{B}_n$ of $\mathbb{C}^n$ and on the cartesian product $\overline{D}^I$ where I is an arbitrary set, maybe finite, infinite denumerable or non-denumerable and $\overline{D}$ is the closed unit disc in $\mathbb{C}$. The class $A(\overline{D}^I)$ contains exactly all functions $f : \overline{D}^I \to \mathbb{C}$ continuous with respect to the product topology on $\overline{D}^I$ and separately holomorphic. We consider sets of uniqueness for $A(\overline{D}^I)$ (respectively for $A(\overline{B}_n)$) to be compact subsets K of T^I (respectively of $\partial\overline{B}_n$) where $T = \partial D$ is the unit circle. If K has positive measure then K is a set of uniqueness. The converse does not hold. Finally, we do a similar study when the uniform convergence is not meant with respect to the usual Euclidean metric in $\mathbb{C}$, but with respect to the chordal metric χ on $\mathbb{C} \cup \{\infty\}$.

1. Introduction

If D is the open unit disc in $\mathbb{C}$ and $\overline{D}$ its closure, then the set of the uniform limits on $\overline{D}$ of polynomials (with respect to the usual Euclidean metric in $\mathbb{C}$) is the well-known disc algebra $A(\overline{D})$; that is the set of all functions $f : \overline{D} \to \mathbb{C}$ continuous on $\overline{D}$ and holomorphic in D.

By Privalov's theorem, a compact set $K \subseteq \partial D = T$ with positive measure is a set of uniqueness for $A(\overline{D})$; that is if $f, g \in A(\overline{D})$ coincide on K, then they coincide on $\overline{D}$. This notion of set of uniqueness is compatible with the ones in [2] and [6]. In fact, the converse also holds: a compact set $K \subseteq T$ is a set of uniqueness for $A(\overline{D})$ if and only if K has a positive measure.

We extend some of the previous results in several complex variables, when $\overline{D}$ is replaced by $\overline{D}^I$ (I arbitrary set even infinite non-denumerable) or the unit ball $\overline{B}_n$ of $\mathbb{C}^n$.

First, we investigate the set of the uniform limits of polynomials. Of course, every polynomial depends on a finite number of variables, even if I is infinite. Thus, we find the classes $A(\overline{D}^I)$ and $A(\overline{B}_n)$ respectively. The class $A(\overline{D}^I)$ contains exactly all functions $f : \overline{D}^I \to \mathbb{C}$ continuous on $\overline{D}^I$ (where $\overline{D}^I$ is endowed with the cartesian topology) which separately as functions of each variable belong to $A(\overline{D})$. The class $A(\overline{B}_n)$

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contains exactly all functions $f : \overline{B}_n \to \mathbb{C}$ continuous on $\overline{B}_n$ and holomorphic in the unit ball B_n. By Hartogs' theorem, if I is finite, this implies that f has a power series development in D^I. In the case where I is infinite we do not need power series expansions on D^I, since separate holomorphicity and continuity are sufficient and necessary for our purposes. Certainly, the previously mentioned results are known in finite dimension and new when the set I is infinite.

We consider T^I $(T = \partial D)$ the distinguished boundary of $\overline{D}^I$ and we also consider compact sets $K \subseteq T^I$ of uniqueness for $A(\overline{D}^I)$. If a compact set $K \subseteq T^I$ has positive measure (with respect to the natural measure on T^I), then K is a set of uniqueness for $A(\overline{D}^I)$. This is based on Privalov's theorem [7] combined with Fubini's theorem. We also give some examples of compact sets $K \subseteq T^I$ with zero measure which are also sets of uniqueness for $A(\overline{D}^I)$, provided that I contains at least two elements.

The boundary $\partial\overline{B}_n$ of the ball of $\mathbb{C}^n$ also carries a natural measure. We prove that if $K \subsetneq \partial\overline{B}_n$ has a positive measure, then K is a set of uniqueness for $A(\overline{B}_n)$. To prove this we do not use an integral representation but we use the corresponding result on the polydisc. If $n \geq 2$ the converse fails.

Next, we repeat all the previous study by replacing the usual Euclidean metric on $\mathbb{C}$ by the chordal metric χ on $\mathbb{C} \cup \{\infty\}$. We investigate the set of uniform limits on $\overline{D}^I$ or $\overline{B}_n$ of polynomials with respect to χ. Thus, we find the classes $\tilde{A}(\overline{D})$, $\tilde{A}(\overline{D}^I)$ and $\tilde{A}(\overline{B}_n)$. The class $\tilde{A}(\overline{D})$ contains $A(\overline{D})$ and is strictly larger, because it contains the function $\frac{1}{1-z}$ which does not belong to $A(\overline{D})$. The precise statement is that a function $f : \overline{D} \to \mathbb{C} \cup \{\infty\}$ belongs to $\tilde{A}(\overline{D})$ if and only if $f \equiv \infty$ or if f is continuous on $\overline{D}$, $f(D) \subseteq \mathbb{C}$ and f is holomorphic in D [1,9].

A compact set $K \subseteq T = \partial\overline{D}$ is a set of uniqueness for $\tilde{A}(\overline{D})$ if and only if it has positive measure. Furthermore, the class $\tilde{A}(\overline{D}^I)$ contains exactly all functions $f : \overline{D}^I \to \mathbb{C} \cup \{\infty\}$ continuous on $\overline{D}^I$ (where $\overline{D}^I$ is endowed with the cartesian topology), which separately for each variable belongs to $\tilde{A}(\overline{D})$.

We consider the notion of a set of uniqueness for $\tilde{A}(\overline{D}^I)$ for compact subsets $K \subseteq T^I (T = \partial D)$ and we prove that if K has positive measure, then it is a set of uniqueness for $\tilde{A}(\overline{D}^I)$. If I contains at least two elements, the converse fails. Since $A(\overline{D}^I) \subseteq \tilde{A}(\overline{D}^I)$, every set of uniqueness for $\tilde{A}(\overline{D}^I)$ is also a set of uniqueness for $A(\overline{D}^I)$. We do not know if the converse holds.

If we endow $A(\overline{D}^I)$ and $\tilde{A}(\overline{D}^I)$ with their natural metrics they become complete metric spaces. In fact, $A(\overline{D}^I)$ is a Banach algebra. Furthermore, the relative topology of $A(\overline{D}^I)$ from $\tilde{A}(\overline{D}^I)$ coincides with the natural topology of $A(\overline{D}^I)$ and $A(\overline{D}^I)$ is open and dense in $\tilde{A}(\overline{D}^I)$.

Finally, we obtain similar results when $\overline{D}^I$ is replaced by $\overline{B}_n$. We notice that in the proof of the main results for $\overline{B}_n$ we use the analogous result for $\overline{D}^I$.

We give a few examples of functions belonging to the previously studied classes. Let $f((z_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \frac{z_j}{j^2}$ for all $(z_j)_{j=1}^{\infty} \in \overline{D}^{\mathbb{N}}$; then $f \in A(\overline{D}^{\mathbb{N}})$.

Let $g(z_1, z_2) = \frac{1}{1-z_1 z_2}$; then $g \in \tilde{A}(\overline{D}^2)$. The previous function f belongs to $A(\overline{D}^{\mathbb{N}})$ and its image is bounded; therefore, if $|c|$ is big enough, the function $c+f(z_1, z_2, \cdots)$ does not vanish at any point of $\overline{D}^{\mathbb{N}}$. Then the function $\frac{c+f(z_1,z_2,\cdots)}{1-z_1}$ also belongs to $\tilde{A}(\overline{D}^{\mathbb{N}})$ and depends on all variables $z_1, z_2, \cdots$. What is a less trivial example of a function belonging to $\tilde{A}(\overline{D}^{\mathbb{N}})$? The class $A(\overline{B}_n)$ is well-known. What are non-trivial examples of functions belonging to $\tilde{A}(\overline{B}_n)$? Such functions are $\omega(z_1, z_2) = \frac{1}{1-z_1}$ and $T(z_1, \cdots, z_n) = \frac{1}{1-z_1^2-z_2^2-\ldots z_n^2}$ with $(z_1, z_2, \cdots, z_n) \in \overline{B}_n$.

An open issue is to study the structure of the element of $\tilde{A}(\overline{D}^I)$ and $\tilde{A}(\overline{B}_n)$. The cases of $A(\overline{D}^I)$ and $A(\overline{B}_n)$ have already been studied if I is a finite set. What happens if I is an infinite set? What is a characterization of the zero sets of elements of $\tilde{A}(\overline{D}^I)$, $\tilde{A}(\overline{B}_n)$ and $A(\overline{D}^I)$, $A(\overline{B}_n)$ when I is infinite? What can be said about compact sets of interpolation for the previous classes? What about peak sets or null-sets? (See [11,12].)

One can see that if $f \in A(\overline{D}^I)$, $f \not\equiv 0$ then $log|f|$ is integrable on T^I with respect to the natural measure. Using a result from [4] one can prove that if $f \in \tilde{A}(\overline{D})$, $f \not\equiv \infty$ then f belongs to the Nevanlinna class;