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Non-spectrality of self-affine measures on the spatial Sierpinski gasket

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ABSTRACT

Let $\mu_{M,D}$ be the self-affine measure corresponding to a diagonal matrix M with entries $p_1, p_2, p_3 \in \mathbb{Z}\backslash\{0, \pm 1\}$ and $D = \{0, e_1, e_2, e_3\}$ in the space $\mathbb{R}^3$, where e_1, e_2, e_3 are the standard basis of unit column vectors in $\mathbb{R}^3$. Such a measure is supported on the spatial Sierpinski gasket. In this paper, we prove the non-spectrality of $\mu_{M,D}$. By characterizing the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$, we obtain that if $p_1 \in 2\mathbb{Z}$ and $p_2, p_3 \in 2\mathbb{Z} + 1$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$. This completely solves the problem on the finiteness or infiniteness of orthogonal exponentials in the Hilbert space $L^2(\mu_{M,D})$.

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix and $D \subset \mathbb{Z}^n$ be a finite digit set of the cardinality $|D|$. Associated with M and D, it is known [7] that here exists a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}, \tag{1.1}$$

such a measure is called *self-affine measure* and is supported on the compact set $T \subset \mathbb{R}^n$, where $T := T(M, D)$ is the *attractor* (or *invariant set*) of the affine iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x + d)\}_{d \in D}$. The measure $\mu_{M,D}$ is called *spectral* if there exists a set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for the Hilbert space $L^2(\mu_{M,D})$. The set Λ is then called a *spectrum* for $\mu_{M,D}$; we also say that $(\mu_{M,D}, \Lambda)$ is a *spectral pair*. The question we are concerned is the spectrality or non-spectrality of $\mu_{M,D}$. This question has its origin in analysis and geometry. It was initiated by Fuglede [6] who investigated which subsets of $\mathbb{R}^n$ with the Lebesgue measures are spectral. In the same paper, Fuglede proposed his famous conjecture on the relationship between the

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spectral set and the translation tile of $\mathbb{R}^n$. This conjecture and its related problems have received much attention in the last few decades (see [10]). In particular, the operator-theoretic approach on these problems initiated by Jorgensen and Pedersen lead the research into the realm of fractals. In terms of fractal measures, it is started with the work of Jorgensen and Pedersen [8,9] who showed that for certain M and D, the measure $\mu_{M,D}$ may be spectral, while for another M and D, the measure $\mu_{M,D}$ may be non-spectral. Subsequently, there are many researches on this question (see [1–3,5,14,15,18,19] and the references cited therein). The previous researches illustrate that the spectrality of $\mu_{M,D}$ requires strict conditions on the Fourier transform $\hat{\mu}_{M,D}$, it has close relation with the problem of finiteness or infiniteness of orthogonal exponentials in the Hilbert space $L^2(\mu_{M,D})$. And for some pairs (M, D), the non-spectrality of $\mu_{M,D}$ is due to the fact that there are at most a finite number of orthogonal exponential functions in the Hilbert space $L^2(\mu_{M,D})$. The present paper will follow the paper [13] to further proving the non-spectrality of self-affine measure $\mu_{M,D}$ on the typical fractal: the spatial Sierpinski gasket $T(M, D)$, where

$$M = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad (p_1, p_2, p_3 \in \mathbb{Z} \setminus \{0, \pm 1\}) \quad \text{and}$$
$$D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{1.2}$$

For such a pair (M, D) in (1.2), the spectrality or non-spectrality of $\mu_{M,D}$ can be summarized as the following Theorem A.

Theorem A. *For the self-affine measure $\mu_{M,D}$ corresponding to (1.2), the following spectrality and non-spectrality hold:*

(i) If $p_1 = p_2 = p_3 = p$ and $p \in 2\mathbb{Z} \setminus \{0\}$, then $\mu_{M,D}$ is a spectral measure;
(ii) If $p_j \in 2\mathbb{Z} \setminus \{0, 2\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a spectral measure;
(iii) If $p_j \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best upper bound.

See [8], [9, Example 7.1], [16], [17, Example 2.9(e)], [4, Theorem 5.1(iii)], [11, Theorem 1], [12]. Also there are two problems on the spectrality of such self-affine measure $\mu_{M,D}$:

Question 1. How about the spectrality of $\mu_{M,D}$ if $p_j \in 2\mathbb{Z} \setminus \{0\}$ $(j = 1, 2, 3)$ and one or two of the three numbers p_1, p_2, p_3 can take the value 2?

Question 2. How about the spectrality of $\mu_{M,D}$ if p_j $(j = 1, 2, 3)$ have different parity?

In a recent paper [13], we settled Question 1 except for the case that two of the three numbers p_1, p_2, p_3 are 2 and the other number is -2, or except for the case that two of the three numbers p_1, p_2, p_3 are -2 and the other number is 2. The spectrality of $\mu_{M,D}$ in the case when $p_j \in 2\mathbb{Z} \setminus \{0\}$ $(j = 1, 2, 3)$ can also be obtained by applying a recent result of [3] and [5]. The answer is that $\mu_{M,D}$ is a spectral measure. So the remaining problem relating to this case is to determine all the spectra for such a measure $\mu_{M,D}$. However, for Question 2, only a little result is known. We summarize the known result as the following Theorem B.

Theorem B. *(See [13].) (i) If M and D are given by (1.2) with $p_1 \in 2\mathbb{Z}$ and $p_2 = p_3 \in 2\mathbb{Z} + 1$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best upper bound; (ii) If M and D are given by (1.2) with any two of the three*