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Periodic solutions of non-isothermal phase separation models with constraint

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ABSTRACT

In this paper, we consider the existence of time periodic solutions to the non-isothermal phase separation models with constraints for the order parameter and the Dirichlet boundary condition for the temperature. We firstly address some approximate problems by the Leray–Schauder fixed point theorem. Then the periodic solutions of the original problem are obtained by establishing some a priori estimates and performing a limiting procedure.

1. Introduction

This paper is concerned with the following phase separation model in the non-isothermal case

$$\begin{cases} (\theta + \lambda_0 w)_t - \triangle\tilde{\alpha} = f(x,t), \quad \tilde{\alpha} \in \alpha(\theta) \quad \text{in} \ \ Q_\omega = \Omega \times (0,\omega), \\ w_t - \triangle\{-\kappa\triangle w + \xi + g(w) - \lambda_0\tilde{\alpha}\} = 0, \quad \xi \in \beta(w) \qquad \text{in} \ \ Q_\omega \end{cases} \tag{1.1}$$

with boundary conditions

$$\frac{\partial w}{\partial \nu} = \frac{\partial}{\partial \nu}(-\kappa\triangle w + g(w) + \xi - \lambda_0\tilde{\alpha}) = 0, \quad \tilde{\alpha} = h(x) \quad \text{on} \ \ \Sigma_\omega = \partial\Omega \times (0,\omega) \tag{1.2}$$

and ω-periodic conditions

$$\theta(x,0) = \theta(x,\omega), \quad w(x,0) = w(x,\omega) \quad \text{in} \ \ \Omega. \tag{1.3}$$

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Here Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\triangle = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$, α, β are the maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, $\kappa > 0$ and $\lambda_0 > 0$ are constants, $g(w)$ is a sufficiently smooth function of $w \in \mathbb{R}$, f and h are given functions on $Q = \Omega \times (0, +\infty)$ and $\Sigma = \partial\Omega \times (0, +\infty)$, respectively.

The system (1.1)–(1.2) arises in the non-isothermal diffusive phase separation of Penrose–Fife type. Actually, the original model of this system was introduced by Penrose and Fife [30] to describe the phase separation in a binary mixture controlled by the temperature, and has so far been studied extensively in various restricted forms. In this context, λ_0 stands for the latent heat, $e = \theta + \lambda_0 w$, θ and w represent the internal energy, (absolute) temperature and the conserved order parameter which describes the local concentration of one of the components, respectively. To reflect this physical sense, it is natural to assume that the range of w is bounded, say the closed interval $[\sigma_*, \sigma^*]$. In fact, the sum $-\Delta w + \beta(w) + g(w) - \lambda_0\tilde{\alpha}$ can be interpreted as the interaction part of a possibly non-smooth free energy. In particular, the term β could coincide with the sub-differential of $\hat{\beta}$ of a double constraint potential, a fairly usual choices are $\hat{\beta} = I_{[\sigma_*,\sigma^*]}$ (the indicator function of $[\sigma_*, \sigma^*]$) or $\hat{\beta}(w) = (w - \sigma_*)\log(w - \sigma_*) + (\sigma^* - w)\log(\sigma^* - w)$, and thus the sub-differential forces w to belong to $[\sigma_*, \sigma^*]$. In this way, system (1.1) is the so-called phase separation model with the constraint. The further explanations of modeling are referred, for instance, to [2,3,5,18–21].

We recall that in the isothermal case, the second equation in (1.1) corresponds to the well-known Cahn–Hilliard equation with the constraint, which has been extensively studied and is now essentially well understood as far as the existence, uniqueness and regularity of solutions. We refer the reader, among a vast literature, to, e.g., [6,8,9,20,24,34]. As the non-isothermal phase separation model, the analogous version of (1.1) has also received much attention during the last decades, see [3,5,8,10,12,13,15,17,19,27,28] and the references therein. It is worth pointing out that in [5,8,13,17,19], as well as in the most of the literature on the subject, third boundary conditions for temperature θ or $\alpha(\theta)$ is essential, the arguments therein are not applicable to the case of the homogeneous Neumann boundary condition [12,15] or Dirichlet boundary condition [10,27,28] and the more delicate analysis is needed. Moreover, the viscosity term $-\mu\Delta w_t\,(\mu > 0)$ is included into the kinetic equation for the conserved order parameter, which plays an important role in obtaining the regularity $w \in W^{1,2}(0, T; L^2(\Omega))$, in the above mentioned papers except [12] where $\alpha(\theta) = -\frac{1}{\theta}$ and the Neumann boundary condition is generally homogeneous. Indeed, only the generalized solution (e, w) of problem (1.1)–(1.3) with initial data (instead of periodic conditions) was obtained in [28], where $e = \theta + \lambda_0 w$, $w \in W^{1,2}(0, T; V_0^*)$, $e \in W^{1,2}(0, T; H^{-1}(\Omega))$ and the first equation in (1.1) is read as $e'(t) + \partial\psi(\theta(t)) \ni f(t) + \Delta h$ in $H^{-1}(\Omega)$ with a proper, l.s.c., convex function ψ on $H^{-1}(\Omega)$.

As a further step toward the comprehension of long-time dynamics such as bifurcation phenomena, it is of interest to study the periodic problem of phase transitions when the heat source f varies periodically. However, to the best of our knowledge, little information is known for the periodic problem of non-isothermal phase transition models [14,26,32]. We recall that the periodic solutions for phase transition models was discussed by Yamazaki [32] and Kumazaki [26] with the help of the general theory (Theorem 2.3.2 of [16]). In fact, they reformulated the phase transition model as an evolution equation of the form

$$(U(t))' + \partial\varphi(U(t)) + G(U(t)) \ni f(t) \quad \text{for a.e.} \quad t \in [0, \omega], \tag{1.4}$$

and thus proved the existence of periodic solutions of phase transition model via the Leray–Schauder fixed point theorem.

Motivated by the aforementioned papers, in this paper, we consider the existence of periodic solutions to (1.1)–(1.3) by the viscosity approach. More precisely, in order to deal with the existence of periodic solutions to (1.1)–(1.3), we introduce a family of approximate problems