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A unified existence theory for evolution equations and systems under nonlocal conditions

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ABSTRACT

We investigate the effect of nonlocal conditions expressed by linear continuous mappings over the hypotheses which guarantee the existence of global mild solutions for functional-differential equations in a Banach space. A progressive transition from the Volterra integral operator associated to the Cauchy problem, to Fredholm type operators appears when the support of the nonlocal condition increases from zero to the entire interval of the problem. The results are extended to systems of equations in a such way that the system nonlinearities behave independently as much as possible and the support of the nonlocal condition may differ from one variable to another.

1. Introduction

This paper deals with the Cauchy problem for functional-differential evolution equations in a Banach space X, with a nonlocal condition expressed by a linear mapping

$$\begin{cases} u'(t) = A(t)\,u(t) + \Phi(u)(t), & \text{for a.a. } t \in [0,a] \\ u(0) = F(u). \end{cases} \tag{1.1}$$

Here $\{A(t)\}_{t\in[0,a]}$ is a family of densely defined linear operators (not necessarily bounded or closed) in the Banach space X generating an evolution operator, Φ is a nonlinear mapping, and F is linear.

Containing the general functional term Φ, our equation is more general than the most studied one given by

$$\Phi(u)(t) = g(t, u_t),$$

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where the function $u_t(s) = u(t+s)$, for $s \in [-r, 0]$, $r > 0$, $t \in [0, a]$ stands for the memory in lots of models for processes with aftereffect (see, e.g. [25]). In particular, it covers evolution equations which are perturbed by a superposition operator Φ,

$$\Phi(u)(t) = f(t, u(t)), \quad t \in [0, a] \tag{1.2}$$

associated to some function $f : [0, a] \times X \to X$, integro-differential equations and equations with modified argument.

In the mathematical modeling of real processes from physics, chemistry or biology, the nonlocal conditions can be seen as feedback controls by which the "sum" of the states of the process along its evolution equals the initial state. The mapping F expressing the nonlocal condition can be linear or nonlinear, of discrete or continuous type. For instance, as a linear mapping, it can be given by a finite sum of multi-point form

$$F(u) = \sum_{k=1}^{m} c_k u(t_k), \tag{1.3}$$

where $0 < t_1 < t_2 < \ldots < t_m \leq a$ and c_k are real numbers. More general, it can be expressed in terms of a Stieltjes integral

$$F(u) = \int_0^a u(t)\, d\phi(t).$$

Nonlocal problems with multi-point conditions and more general with linear and nonlinear nonlocal conditions were discussed in the literature by various approaches. We refer the reader to the papers [1,4–11,14,17,18,20,24,26] and the references therein.

As it was first remarked in [6], it is important to take into consideration the *support* of the nonlocal condition, that is the minimal closed subinterval $[0, a_F]$ of $[0, a]$ with the property

$$F(u) = F(v) \text{ whenever } u = v \text{ on } [0, a_F]. \tag{1.4}$$

This means that the mapping F only depends on the restrictions of the functions from $C([0, a]; X)$, to the subinterval $[0, a_F]$. The case $a_F = 0$ recovers the classical Cauchy problem, while the case $a_F = a$ corresponds to a *global* nonlocal condition dissipated over the entire interval $[0, a]$ of the problem. When $0 < a_F < a$, we say that the nonlocal condition is *partial*. As we shall see, moving a_F from 0 to a, we realize a progressive transition from Volterra to Fredholm nature of the equivalent integral equation.

The support problem is even more interesting in case of a system of equations in n unknown functions $u_1, u_2, \ldots, u_n$, when a nonlocal condition is expressed by a linear mapping $F = F(u_1, u_2, \ldots, u_n)$. In this case, we may speak about the *support of F with respect to each of the variables*. The notion is introduced in this paper for the first time, and together with the vectorial method that is used, allows us to localize independently each component u_i of a solution $(u_1, u_2, \ldots, u_n)$.

In addition, as an other original feature of our study, the localization of a solution, and in case of systems, of each of the solution components, is realized in a *tube*, i.e. a set of the form

$$\{(t, u) : \; t \in [0, a], \; u \in X, \; |u| \leq R(t)\},$$

of a time-depending radius $R(t)$. In a physical interpretation, this means that the variation of a quantity $u(t)$ is allowed to be nonuniformly larger or smaller during the evolution, as prescribed by function $R(t)$.