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Cancellation properties of composition operators on Bergman spaces ☆

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A B S T R A C T

The compact difference of two composition operators on the Bergman spaces over the unit disc is characterized in [11] in terms of certain cancellation property of the inducing maps at every "bad" boundary points, which make each single composition operator not to be compact. In this paper, we completely characterize the compactness of a linear combination of three composition operators on the Bergman space. As one consequence of this characterization, we show that there is no cancellation property for the compactness of double difference of composition operators. More precisely, we show that if φ_i are distinct and none of C_{φ_i} is compact, then $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ is compact if and only if both $(C_{\varphi_1} - C_{\varphi_2})$ and $(C_{\varphi_3} - C_{\varphi_1})$ are compact.

1. Introduction

A holomorphic self-map φ of the unit disc $\mathbf{D}$ in complex plane $\mathbf{C}$ induces a composition operator $C_\varphi : H(\mathbf{D}) \to H(\mathbf{D})$ defined by

$$C_\varphi f := f \circ \varphi,$$

where $H(\mathbf{D})$ is the class of all holomorphic functions on $\mathbf{D}$. An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to [5] and [17] for various aspects on the theory of composition operators acting on holomorphic function spaces.

On the unit disc, every composition operator is bounded on the weighted Bergman spaces or the Hardy spaces due to Littlewood's subordination principle and much effort has been expended on characterizing those holomorphic maps which induce compact composition operators. Early result of Shapiro and Taylor

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[19] in 1973 showed the non-existence of the angular derivative of the inducing map at any point of the boundary of the unit disc is a necessary condition for the compactness of the composition operator on the Hardy space $H^2(\mathbf{D})$. Later, MacCluer and Shapiro [10] proved that this condition is a necessary and sufficient condition for the compactness of composition operators on the weighted Bergman spaces $A^p_\alpha(\mathbf{D})$. Using the Nevanlinna counting function, Shapiro [16] completely characterized those φ which induce compact composition operators on the Hardy space $H^2(\mathbf{D})$.

With the basic questions such as boundedness and compactness settled, it is natural to look at the topological structure of the composition operators in the operator norm topology and this topic is of continuing interests in the theory of composition operators. Berkson [1] focused attention on topological structure with his isolation result on $H^p(\mathbf{D})$ in 1981 which was refined later by Shapiro and Sundberg [18], and MacCluer [9]. In [18], Shapiro and Sundberg posed a question: Do the composition operators on $H^2(\mathbf{D})$ that differ from C_φ by a compact operator form the component of C_φ in the operator norm topology? While the same question was answered positively on the weighted Bergman spaces [9], this turned out to be not true on the Hardy space [2,6,12]. In relation to the study of the topological structures, the difference or the linear sum of composition operators on various settings has been a very active topic [3,4,8,9,12–15]. On the Bergman space over the unit disc, a complete characterization of when the difference of two composition operators is compact is given by Moorhouse [11], while the same problem for the Hardy space still remains open. The essence of Moorhouse's compact difference characterization is certain cancellation property between the symbol maps on every boundary points which make each inducing composition operator not to be compact. This raises a natural question: Is it possible that the double difference of composition operators is compact while both differences are not compact? More precisely, one can raise the following question:

Can $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ be compact while both $(C_{\varphi_1} - C_{\varphi_2})$ and $(C_{\varphi_3} - C_{\varphi_1})$ are not compact?

In this paper, we completely characterize the compactness of a linear combination of three composition operators, and as an application we show that the double difference cancellation cannot occur on the weighted Bergman spaces over the unit disc.

For $0 < p < \infty$ and $\alpha > -1$, the α-weighted Bergman space $A^p_\alpha(\mathbf{D})$ is the space of all $f \in H(\mathbf{D})$ such that the "norm"

$$\|f\|_{A^p_\alpha} := \left(\int_{\mathbf{D}} |f(z)|^p \, dA_\alpha(z) \right)^{1/p}$$

is finite, where dA_α is the normalized area measure on $\mathbf{D}$. We let $A^p(\mathbf{D}) = A^p_0(\mathbf{D})$. As is well-known, for each $\alpha > -1$ the space $A^p_\alpha(\mathbf{D})$ equipped with the norm above is a Banach space for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|^p_{A^p_\alpha}$.

Throughout the paper we assume $\varphi_j : \mathbf{D} \to \mathbf{D}$ is holomorphic ($j \in \mathbf{N}$) and $\varphi_i \neq \varphi_j$ if $i \neq j$. We also use the following notation throughout the paper:

$$F_i = \{\zeta \in \partial\mathbf{D} : \varphi_i \text{ has a finite angular derivative at } \zeta\} \tag{1.1}$$

and

$$\rho_{ij}(z) = \left| \frac{\varphi_i(z) - \varphi_j(z)}{1 - \overline{\varphi_i(z)}\varphi_j(z)} \right|. \tag{1.2}$$

Our main results are the following:

Theorem 1.1. *Let $0 < p < \infty$ and $\alpha > -1$. Let $a_i \in \mathbf{C} \setminus \{0\}$ and assume C_{φ_i} is not compact on $A^p_\alpha(\mathbf{D})$ for each $i = 1, 2, 3$. If $T := \sum_{i=1}^3 a_i C_{\varphi_i}$ is compact on $A^p_\alpha(\mathbf{D})$, then one of the following holds:*