J. Math. Anal. Appl. 432 (2015) 114–128

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Estimates for the moments of Bernstein polynomials

José A. Adell [a], Jorge Bustamante [b], José M. Quesada [c,*]

[a] *Department of Statistics, Universidad de Zaragoza, Spain*
[b] *Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, Mexico*
[c] *Department of Mathematics, Universidad de Jaén, Spain*

A R T I C L E I N F O

Article history:
Received 18 February 2015
Available online 15 May 2015
Submitted by P. Nevai

Keywords:
Bernstein polynomials
Bernstein moments
Rate of convergence
Voronovskaja's formula

A B S T R A C T

We give various explicit estimates for the moments and the absolute moments of Bernstein polynomials. Asymptotically, such estimates are not far from optimality, specially for high moments. As an application, generalized Voronovskaja's formulae for Bernstein polynomials are discussed.

1. Introduction and main results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every function $f : [0, 1] \to \mathbb{R}$ and any $n \in \mathbb{N}$, the nth Bernstein polynomial of f is defined by

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, \qquad x \in [0, 1]. \tag{1}$$

In several problems related with the behaviour of Bernstein polynomials, it is necessary to estimate the moments and the absolute moments respectively defined by

$$\mu_{n,k}(x) = B_n\Big((e_1 - x)^k, x\Big), \quad n \in \mathbb{N}, \quad x \in [0, 1], \quad k \in \mathbb{N}_0, \tag{2}$$

and

$$M_{n,r}(x) = B_n\Big(|e_1 - x|^r, x\Big), \quad n \in \mathbb{N}, \quad x \in [0, 1], \quad r > 0, \tag{3}$$

* Corresponding author.
E-mail addresses: adell@unizar.es (J.A. Adell), jbusta@fcfm.buap.mx (J. Bustamante), jquesada@ujaen.es (J.M. Quesada).

http://dx.doi.org/10.1016/j.jmaa.2015.05.020
0022-247X/

where $e_1(t) = t$. For instance, it is known [6, Th. 2.3] that, for $0 < \alpha \le 1$,

$$\sup_{f \in Lip^\alpha[0,1]} |B_n(f,x) - f(x)| = M_{n,\alpha}(x),$$

where $Lip^\alpha[0,1]$ is the family of all functions $f : [0,1] \to \mathbb{R}$ such that

$$|f(x) - f(y)| \le |x-y|^\alpha, \qquad x, y \in [0,1].$$

For multivariate Bernstein polynomials, we refer the reader to [10]. On the other hand, Bernstein [1] was the first to note that the central moments $\mu_{n,k}$ are important to obtain Voronovskaja's type theorems. In this respect, the following inequalities are known

$$|\mu_{n,k}(x)| \le \frac{k!}{n^{k/2}} \exp\Big(x(1-x)\Big), \qquad \max\{x, 1-x\} \le \frac{1}{\sqrt{n}}, \qquad \text{[1, Bernstein]},$$

$$|\mu_{n,k}(x)| \le \frac{k!\, n^{1/5}}{(n \ln n)^{k/2}}, \qquad x \in [0,1],\ \ n \ge 2, \qquad \text{[16, Veselinov]},$$

$$\mu_{n,2k}(x) \le \frac{A_k}{n^k}, \qquad x \in [0,1], \qquad \text{[8, p. 15, Lorentz]},$$

$$\mu_{n,2k}(x) \le A_k \left(\frac{x(1-x)}{n}\right)^k, \qquad B > 0, \frac{B}{n} \le x \le 1 - \frac{B}{n}, \qquad \text{[2, Ditzian]},$$

$$|\mu_{n,k+1}(x)| \le \frac{A_k x(1-x)}{n^k}, \qquad nx(1-x) \le 1, \qquad \text{[3, Gavrea–Ivan]}$$

$$M_{n,k}(x) \le A_k \left(\max\left\{\frac{1}{n}, \sqrt{\frac{x(1-x)}{n}}\right\}\right)^k, \qquad x \in [0,1], \qquad \text{[9, Lorentz]},$$

$$M_{n,k}(x) \le A_k \left(\frac{x(1-x)}{n}\right)^{k/2}, \qquad nx(1-x) \ge 1, \qquad \text{[15, Telyakovskii]},$$

where $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and A_k is an unspecific constant only depending upon k.

The aim of this paper is to improve the aforementioned inequalities by providing explicit estimates both for the moments and the absolute moments of Bernstein polynomials. As an application, we discuss in the final section generalized Voronovskaja's formulae for such polynomials. The following are our main results.

Theorem 1. *For any $n \in \mathbb{N}$, $x \in [0,1]$, and $r > 0$, we have*

$$M_{n,r}(x) \le 2\,\Gamma\left(\frac{r}{2} + 1\right) \frac{1}{n^{r/2}},$$

where $\Gamma(\cdot)$ stands for the gamma function.

Theorem 2. *In the setting of Theorem 1, for any $\delta \in (0,1)$, one has*

$$M_{n,r}(x) \le 2\left(\left(\frac{2x(1-x)}{1-\delta}\right)^{r/2} \Gamma\left(\frac{r}{2}+1\right) + \left(\frac{2}{\delta(1-\delta)}\right)^r \frac{\Gamma(r+1)}{n^{r/2}}\right) \frac{1}{n^{r/2}}.$$

In particular, we have for any $k \in \mathbb{N}$

$$\mu_{n,2k}(x) \le 2\left(\left(\frac{2x(1-x)}{1-\delta}\right)^k k! + \left(\frac{2}{\delta(1-\delta)}\right)^{2k} \frac{(2k)!}{n^k}\right) \frac{1}{n^k}.$$