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On the rate of convergence of discretely defined operators

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ABSTRACT

We prove that a result of Tachev concerning the optimal rate of convergence of the classical Bernstein operators remains valid for the class of discretely defined positive linear operators preserving constants.

1. Introduction

For any non-negative integer n, the Bernstein basis polynomials of degree n are defined as

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

The Bernstein [3] operator $B_n : C[0,1] \to C[0,1]$ is defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \qquad x \in [0,1].$$

The rate of convergence of sequences of positive linear operators is a main problem in approximation theory. A key tool in this direction is the Voronovskaja theorem [17] and its generalization given by Bernstein:

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Theorem 1. *(See Bernstein [2].) If $q \in \mathbb{N}$ is even and $f \in C^q[0,1]$, then,*

$$\lim_{n\to\infty} n^{q/2}\left(B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n((\cdot - x)^r; x)\frac{f^{(r)}(x)}{r!}\right) = 0,$$

uniformly on $[0,1]$.

Let $L_n\colon C[0,1] \to C[0,1]$, $n = 1, 2, \ldots$, be a sequence linear operators. Throughout the paper we will use the condensed notation:

$$R(L_n, f, q, x) := \left| L_n(f;x) - \sum_{i=0}^{q} L_n((\cdot - x)^i; x)\frac{f^{(i)}(x)}{i!}\right|.$$

Generalizations of the classical Voronovskaja's theorem [17] related to the Bernstein polynomials were intensively studied also by Mamedov [11], Sikkema and van der Meer [12], and in many subsequent papers that deal with this topic (see, e.g., [9,14]).

In [14, Theorem 2], Tachev extended the classical result of Bernstein (Theorem 1) to odd values of q.

In [7] we gave an affirmative answer to a conjecture of Tachev [14, Eq. (2.15)] related to the Bernstein Theorem 1.

In [8] we proved that the Bernstein Voronovskaja-type property (Theorem 1) satisfied by the Bernstein operators remains valid in general for all sequences of positive linear approximation operators.

In this paper we prove that the result of Tachev [14, Thm. 3] (rewritten below) concerning the optimal rate of convergence of the classical Bernstein operators remains valid for the class of all discretely defined positive linear operators preserving constants.

Theorem 2. *(See [14, Thm. 3, Tachev (2012)].) For any $q \in \mathbb{N}$ and $\beta > 0$ there exists a function $f_0 \in C^q[0,1]$ and a point $x_0 \in (0,1)$ such that*

$$\lim_{n\to\infty} n^{q/2+\beta}\, R(B_n, f_0, q, x_0) = \infty.$$

2. Auxiliary results

In 1730, Abraham De Moivre gave a simple closed form expression for the mean absolute deviation of the binomial distribution (see [5]) which we will prove below in terms of Bernstein polynomials:

We start from the recurrence relations

$$\left(x - \frac{k}{n}\right) p_{n,k}(x) = x(1-x)\Big(p_{n-1,k}(x) - p_{n-1,k-1}(x)\Big), \tag{2.1}$$

$n = 1, 2, \ldots$, $k = 0, 1, \ldots, n$, where $p_{n,-1} = p_{n,n+1} := 0$ (see, e.g., [4, Chapter 10, Eq. (2.1)]). From (2.1) we deduce

$$\sum_{k=0}^{s} p_{n,k}(x)\left(x - \frac{k}{n}\right) = x(1-x)p_{n-1,s}(x) = x\left(1 - \frac{s}{n}\right)p_{n,s}(x), \tag{2.2}$$

$s = 0, 1, \ldots, n$. By using (2.2), we obtain: