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On the stability of a periodic solution of distributed parameters biochemical system

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ABSTRACT

This paper studies the stability of periodic solutions of distributed parameters biochemical system with periodic input $S_{in}(t)$. We prove that if $S_{in}(t)$ is periodic then the system has a periodic solution that is input to state stable when small perturbations are acting on the input concentration $S_{in}(t)$.

1. Introduction

The aim of this work is to prove the existence and stability of periodic solutions of a model describing a biochemical reactor with periodic input $S_{in}(t)$. Periodic solutions arise in many bioengineering systems because of the often periodically time varying environments. In the last decades, the existence of such periodic solutions has been extensively investigated by many authors to understand oscillations observed in many chemostat experiments (see e.g. [19,10,11,14], and references therein). The chemostat is an experimental device used to understand the dynamics of biological, biochemical or ecological systems and in which the components of the systems are only time varying. Parallel to chemostat systems, the dynamical analysis and control of tubular (bio)chemical reactors have also motivated many research activities over the last decades (see e.g. [1–4,7,18], etc. and references therein). These studies are mostly focused on existence and asymptotic behavior of state trajectories, control and observability of the systems, in which a linearization of the system is the underlying tool. Following the ideas in the theoretical and experimental results in chemostat studies, recently Drame et al. [5,6] studied the existence of periodic and almost periodic solutions

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of distributed parameters biochemical systems. It was shown that periodic solutions of a time delay system exist with a constant input S_{in} in [5], and with time varying input $S_{in}(\cdot)$ in [6], but both studies lack a stability analysis. Pilyugin and Waltman [16] studied a reaction–diffusion system describing an unstirred chemostat and prove the existence of periodic solutions based on a system reduction technique. However, the system in [16] is monotone and the method cannot be applied to the systems considered here or in [5,6].

It is natural to assume in a tubular biochemical reactor's model that the input nutrient concentration $S_{in}(t)$ is time dependent and periodic in the time t. The dynamical system under consideration in the current paper uses this assumption in a model involving a diffusion-transport partial differential equation coupled with a nonlinear ordinary differential equation. The coupling term involves both the biomass and substrate. The justification of the model is derived from work performed on anaerobic digestion in the pilot fixed bed reactor of the LBE-INRA in Narbonne (France) and is validated on the process (see [1,17]). Our main result involves a Lyapunov functional technique that analyzes and proves that if we replace $S_{in}(t)$ by $S_{in}(t) + a(t)$, where $a(t)$ is a small perturbation, then this will have small effect on the periodic solution. The paper is organized as follows. The model, background, and preliminary results are given in Section 2. Section 3 introduces an auxiliary system and gives an existence result for a solution. Section 4 is devoted to the existence problem of a periodic solution of the main system under study. The main new results are contained in Section 5 where a stability analysis is presented.

2. Notation, the model and preliminary results

2.1. Notation and Schauder's Fixed Point Theorem

The notation is standard and will be simplified whenever no confusion can arise from the context. The Euclidean norm of vectors of any dimension is denoted $|\cdot|$. For a function $\varphi \in L^2(0,1)$, the L^2 norm is $\|\varphi\|_{L^2} = \sqrt{\int_0^1 |\varphi(m)|^2 dm}$. We let $\mathcal{Z} = C[0,1] \times C[0,1]$.

The set of modulus functions is denoted by $\mathcal{K}_\infty$ and consists of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$ satisfying (i) $\gamma(0) = 0$, (ii) $\gamma(.)$ is strictly increasing, and (iii) $\gamma(.)$ is unbounded.

We recall the Schauder's Fixed Point Theorem [12, p. 126], which will be later invoked to provide existence result. Let $\mathcal{X}$ be Banach space and $\mathcal{D} \subseteq \mathcal{X}$. Recall that a completely continuous function $A(.) : \mathcal{D} \to \mathcal{X}$ is a continuous function that maps bounded sets into relatively compact ones.

Theorem 2.1 *(Schauder's Theorem). Suppose that $\mathcal{D}$ is a closed bounded convex subset of a Banach space $\mathcal{X}$ and $A(.) : \mathcal{D} \to \mathcal{X}$ is a completely continuous function with $A(\mathcal{D}) \subseteq \mathcal{D}$. Then there is a point $z \in \mathcal{D}$ such that $Az = z$.*

2.2. The model

Applying the mass balance principles to the limiting substrate concentration $S(t,z)$ and the living biomass concentration $X(t,z)$ leads to the following dynamical system:

$$\begin{aligned} \frac{\partial S}{\partial t}(t,z) &= d\frac{\partial^2 S}{\partial z^2}(t,z) - q\frac{\partial S}{\partial z}(t,z) - k\mu(S(t,z),X(t,z))X(t,z)\ , \\ \frac{\partial X}{\partial t}(t,z) &= -k_d X(t,z) + \mu(S(t,z),X(t,z))X(t,z)\ , \end{aligned} \tag{2.1}$$

with the boundary conditions

$$d\frac{\partial S}{\partial z}(t,0) - qS(t,0) + qS_{in}(t) = 0 \quad \text{and} \quad \frac{\partial S}{\partial z}(t,L) = 0 \quad \text{for all} \quad t \geq 0 \tag{2.2}$$