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Generalized solution to a system of conservation laws which is not strictly hyperbolic

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ABSTRACT

In this paper we study a non-strictly hyperbolic system of conservation laws when viscosity is present and when viscosity is zero, which has been studied in [12]. We show the existence and uniqueness of the solution in the space of generalized functions of Colombeau for the viscous problem and construct a solution to the inviscid system in the sense of association. Also we construct a solution using shadow wave approach [17] and Volpert product which was partially determined as vanishing viscosity limit in [12].

1. Introduction

Conservation laws in applications are sometime not strictly hyperbolic. The classical theory of Lax [15] and Glimm [6] does not apply in this case. In general such systems do not admit distributional solutions. As a product of distributions arises, one cannot expect solutions in the space of distributions. The ideal space where one should search for solutions is the Colombeau algebra of generalized functions. For details, see Colombeau [3–5] and Oberguggenberger [18].

Our interest is to study the system of inviscid partial differential equations, namely;

$$u_t + (\frac{u^2}{2})_x = 0, \quad v_t + (uv)_x = 0$$
$$w_t + (\frac{v^2}{2} + uw)_x = 0, \quad z_t + (vw + uz)_x = 0 \tag{1.1}$$

and viscous regularization of (1.1) with coefficient of viscosity being a generalized constant γ,

$$u_t + (\frac{u^2}{2})_x = \frac{\gamma}{2} u_{xx}, \quad v_t + (uv)_x = \frac{\gamma}{2} v_{xx}$$

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$$w_t + (\frac{v^2}{2} + uw)_x = \frac{\gamma}{2}w_{xx}, \quad z_t + (vw + uz)_x = \frac{\gamma}{2}z_{xx}, \tag{1.2}$$

with initial conditions

$$(u(x,0), v(x,0), w(x,0), z(x,0)) = (u_0, v_0, w_0, z_0) \tag{1.3}$$

given by generalized functions of Colombeau.

The system which we are considering here is the $n = 4$ case of the following system:

$$(u_j)_t + \sum_{i=1}^{j}(\frac{u_i u_{j-i+1}}{2})_x = \frac{\epsilon}{2}(u_j)_{xx},\ j = 1, 2, \dots n, \tag{1.4}$$

with initial condition

$$u_j(x, 0) = u_{j0}(x),\ \ j = 1, 2, \dots, n, \tag{1.5}$$

where $\epsilon > 0$ is a small parameter. This system (1.4) is introduced in [13]. It is shown there that the system (1.4) can be linearized by using a generalized Hopf–Cole transformation, this in turn gives explicit formula for $u_i, i = 1, 2...n$.

The corresponding inviscid system

$$(u_j)_t + \sum_{i=1}^{j}(\frac{u_i u_{j-i+1}}{2})_x = 0,\ j = 1, 2, \dots n \tag{1.6}$$

is not strictly hyperbolic as it has repeated eigenvalues, i.e., $\lambda_i(u_1, u_2, \dots, u_n) = u_1$, for $i = 1, 2..., n$. It is well known that the above inviscid system does not have smooth global solution, even if the initial data (1.5) is smooth; one has to seek a solution in a weak sense and weak solutions are not unique. Additional conditions are required to pick the unique physical solution. Vanishing viscosity method is one of the ways to select the physical weak solution of (1.4). That is, the solution of the inviscid system is constructed as the limit ϵ goes to zero of solutions $u_j^\epsilon(x, t)$ of (1.4), with suitable initial conditions. This was successfully carried out for the cases $n = 1$ and $n = 2$ for general initial data. For $n = 3$, only partial results are available. In fact it was observed in [10] that the order of singularity of vanishing viscosity limit of solutions of (1.4) increases as n increases.

More precisely, when $n = 1$, with $u = u_1$, (1.4) is the celebrated Burgers equation,

$$u_t + (\frac{u^2}{2})_x = \frac{\epsilon}{2}u_{xx},$$

which was explicitly solved for the initial value problem by Hopf [8] and Cole [2]. Hopf [8] showed that the vanishing viscosity limit of its solution with a given bounded measurable initial data is a bounded measurable and locally of bounded variation, which is the weak entropy solution to the inviscid Burgers equation

$$u_t + (\frac{u^2}{2})_x = 0.$$

Viscous and inviscid Burger's equation in Colombeau setting have been studied in [1]. For $n = 2$, with $u = u_1, v = u_2$ the system (1.4) becomes

$$u_t + (\frac{u^2}{2})_x = \frac{\epsilon}{2}u_{xx},\ \ v_t + (uv)_x = \frac{\epsilon}{2}v_{xx},$$