J. Math. Anal. Appl. 432 (2015) 324–337

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

Gerhard Kristensson

*Department of Electrical and Information Technology, Lund University, P.O. Box 118, SE-221 00 Lund, Sweden*

A R T I C L E I N F O

*Article history:*
Received 26 May 2015
Available online 2 July 2015
Submitted by B. Kaltenbacher

*Keywords:*
Erdélyi operator
Spherical waves
Scattering by random objects
Pair correlation function

A B S T R A C T

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.


## 1. Introduction

In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in *e.g.* [3–8,10,11,13,16–19,21–25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see *e.g.* [12,14,20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9,18,20]

$$I_l(z) = \frac{k^2}{2\pi} \iint\limits_{\mathbb{R}^2} H(r-a) h_l^{(1)}(kr) P_l(\cos\theta)\,\mathrm{d}x\,\mathrm{d}y, \qquad z \in \mathbb{R} \tag{1}$$

where $H(x)$ denotes the Heaviside function, and $h_l^{(1)}(kr)$ and $P_l(x)$ denote the spherical Hankel function and the Legendre polynomial of order $l$, respectively. We have also adopted the spherical coordinates,

*E-mail address:* Gerhard.Kristensson@eit.lth.se.

http://dx.doi.org/10.1016/j.jmaa.2015.06.047

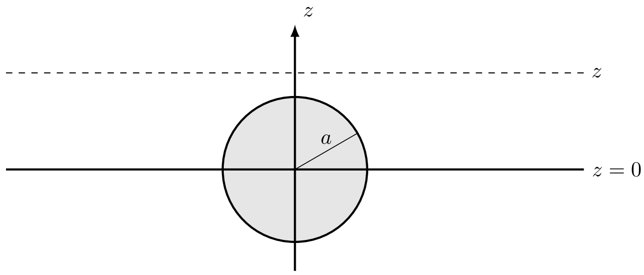


**Fig. 1.** The geometry of the integration domain – the plane $z$ = constant (dotted line), and the exclusion volume – the sphere of radius $a$ located at the origin (in gray).

$r = \sqrt{x^2 + y^2 + z^2}$ and $\theta$ ($\cos\theta = z/r$), and the wave number $k$. The domain of integration is the plane $z =$ constant, excluding the sphere of radius $a > 0$ at the center, see Fig. 1. For a given value of $|z| \leq a$, the radius of the excluded circle is $\sqrt{a^2 - z^2}$. For $|z| \geq a$ the integration is the entire $x$–$y$ plane. This integral, for a given $a > 0$, is a non-trivial function of $z \in \mathbb{R}$. To ensure convergence of the integral at infinity, we assume the wave number $k$ has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of $z$ and the index $l = 0, 1, 2, \ldots$, is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral $I_l(z)$ is developed in Sections 2 and 3. The indefinite Fourier transform of $I_l(z)$ is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.

## 2. The integral $I_l(z)$

Rewrite the integral $I_l(z)$ in (1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)}\left(k\sqrt{\rho^2 + z^2}\right) P_l\left(z/\sqrt{\rho^2 + z^2}\right) \rho \, \mathrm{d}\rho, \quad z \in \mathbb{R} \tag{2}$$

where

$$h(z) = \begin{cases} \sqrt{a^2 - z^2}, & -a \leq z \leq a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials, $P_l(-x) = (-1)^l P_l(x)$, we see that also $I_l(-z) = (-1)^l I_l(z)$. Thus, it suffices to evaluate the integral for $z > 0$. In particular, $I_l(0) = 0$, if $l$ is an odd integer. From (2) we also easily compute the integral for $l = 0$, *viz.*

$$I_0(z) = \begin{cases} \mathrm{e}^{-\mathrm{i}kz}, & z \leq -a \\ \mathrm{i}kah_0^{(1)}(ka) = \mathrm{e}^{\mathrm{i}ka}, & -a \leq z \leq a \\ \mathrm{e}^{\mathrm{i}kz}, & z \geq a \end{cases}$$

### *2.1. Solution outside the interval* $[-a, a]$

In the interval $z > a$, the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], *i.e.*, for a general value of $z \neq 0$