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# Liouville type results for semi-stable solutions of the weighted Lane–Emden system

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ABSTRACT

We examine the weighted Lane–Emden system

$$\begin{cases} -\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} v^p, \\ -\Delta v = (1+|x|^2)^{\frac{\alpha}{2}} u^q, \end{cases} \quad \text{in } \mathbb{R}^N,$$

where $1 < p \le q$ and $\alpha > 0$, and the weighted Lane–Emden equation

$$-\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} u^p, \quad \text{in } \mathbb{R}^N,$$

where $p > \frac{4}{3}$ and $\alpha > 0$. We prove that the system (equation) does not have the classical positive semi-stable solution in dimension $N < 2 + \frac{(4+2\alpha)(q+1)}{pq-1}\tau_0^+$ $\left(N < 2 + \frac{2(2+\alpha)}{p-1}\left(p + \sqrt{p^2-p}\right)\right)$. In particular, there is no positive semi-stable solution of the weighted Lane–Emden system if $N \le 10 + 4\alpha$ and $2 \le p \le q$.


## 1. Introduction

We consider the following weighted Lane–Emden system

$$\begin{cases} -\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} v^p, \\ -\Delta v = (1+|x|^2)^{\frac{\alpha}{2}} u^q, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $1 < p \le q$ and $\alpha > 0$, and the weighted Lane–Emden equation

$$-\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} u^p, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

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where $p > 1$ and $\alpha > 0$. We will establish Liouville type results, i.e., the nonexistence of the classical positive semi-stable solution for (1.1) and (1.2) in $\mathbb{R}^N$.

For the second order Lane–Emden equation, Farina [8] gave a complete classification of stable solution and finite Morse index solution for the nonlinear problem

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{1.3}$$

where $\Omega$ is an unbounded domain with $N \geq 2$. Farina's approach made a delicate application of the classical Moser's iteration. There exist many excellent papers to use his approach to consider the Hardy–Hénon equation and the weighted nonlinear elliptic equations. We refer to [3,5,13,19] and the references therein.

However, Farina's approach may fail to obtain the complete classification for stable solution and finite Morse index solution of the biharmonic equation

$$\Delta^2 u = |u|^{p-1}u, \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain or the entire space. Recently, Dávila–Dupaigne–Wang–Wei [4] provided a new method—a combination of monotonicity formula and blowing down sequence—to deal with the complete classification. Adopting the monotonicity formula approach, Du–Guo–Wang [6] and Hu [12] respectively obtained a complete classification of stable solution and finite Morse index solution of the second order weighted nonlinear elliptic equations and the fourth order Hénon equation $\Delta^2 u = |x|^\alpha |u|^{p-1}u$.

We note that the above several approaches do not work well with some elliptic equations and systems (e.g., the nonhomogeneous weighted elliptic system (1.1), biharmonic equation with negative exponent). A new approach which is obtained by Cowan–Ghoussoub [2] and Dupaigne–Ghergu–Goubet–Warnault [7] independently can handle the Lane–Emden system and biharmonic equation with negative exponent. Thus one combine the second order stability criterion with bootstrap iteration.

For the general equation or system with $\alpha \neq 0$, the Liouville property is less understood and is more delicate to deal with than $\alpha = 0$. Using Farina's approach, Fazly proved Liouville type theorem of the weighted Lane–Emden equation (1.2). That is,

**Theorem A.** *(See [9, Theorem 2.3].) If $(u, v)$ is a $C^2(\mathbb{R}^N)$ nonnegative entire semi-stable solution of (1.2) with $p \geq 2$, and the space dimension satisfies*

$$N < 2 + \frac{2(2+\alpha)}{p-1}\left(p + \sqrt{p^2 - p}\right), \tag{1.4}$$

*then $u$ is the trivial solution.*

Moreover, he [9] also raised an open question:

*How can one establish Liouville type theorem for positive semi-stable solution of (1.1) with the value of $p, q > 1$?*

Inspired by the ideas in [1,10,11], we adopt the new approach—a combination of second order stability, Souplet's inequality [18] and bootstrap iteration—to establish Liouville type theorems for the semi-stable solutions of (1.1) and (1.2), and present an affirmative answer to the above open question.

We first define two parameters which play an important role in stating our main results. Fix $1 \leq p \leq q$, we denote

$$\tau_0^- := \sqrt{\frac{pq(p+1)}{q+1}} - \sqrt{\frac{pq(p+1)}{q+1} - \sqrt{\frac{pq(p+1)}{q+1}}},$$