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Consequences of universality among Toeplitz operators

Carl C. Cowen [a,*], Eva A. Gallardo-Gutiérrez [b]

[a] *Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis, Indianapolis, IN 46202, USA*
[b] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid and ICMAT, Plaza de Ciencias Nº 3, 28040 Madrid, Spain*

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ABSTRACT

The *Invariant Subspace Problem* for Hilbert spaces is a long-standing question and the use of universal operators in the sense of Rota has been one tool for studying the problem. The best known universal operators have been adjoints of analytic Toeplitz operators or unitarily equivalent to them. We present many examples of Toeplitz operators whose adjoints are universal operators and exhibit some of their common properties. Some ways in which the invariant subspaces of these universal operators interact with operators in their commutants are given. Special attention is given to the closed subalgebra, not always the zero algebra, of compact operators in their commutants. Finally, three questions connecting shift invariant subspaces and invariant subspaces of analytic Toeplitz operators are raised. Positive answers for both of the first two imply the existence of non-trivial invariant subspaces for every bounded operator on separable Hilbert spaces of dimension two or more.

1. Introduction

The *Invariant Subspace Problem* is an important question in functional analysis but remains unsolved in the context of separable, infinite dimensional Hilbert spaces. Radjavi and Rosenthal's classic book [21] and the recent monograph by Chalendar and Partington [3] are excellent resources for both references and techniques developed in order to solve this and related problems.

One approach to this problem has been the use of universal operators in the sense of Rota, a class of operators whose structure is rich enough to model every operator on a separable infinite dimensional Hilbert space. There are several well-known examples of universal operators in the literature, and most of these are adjoints of analytic Toeplitz operators or operators that have a reducing subspace on which the operator is universal and unitarily equivalent to the adjoint of an analytic Toeplitz.

* Corresponding author.
E-mail addresses: ccowen@math.iupui.edu (C.C. Cowen), eva.gallardo@mat.ucm.es (E.A. Gallardo-Gutiérrez).

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In this work, we will examine the class of adjoints of analytic Toeplitz operators that are universal in the sense of Rota and greatly extend the class of known examples. We will then develop a strategy for attacking the *Invariant Subspace Problem* based on using operators that commute with the universal operators along with the structure of analytic Toeplitz operators. There are certain reductions in the problem that this makes possible and some of these results are stated as alternatives. Of course, the main difficulty with these results is that for most specific bounded linear operators, it is not clear which of the alternatives holds!

The invariant subspaces of the unilateral shift T_z acting on the classical Hardy space H^2, and therefore, of the backward shift are well known. These ideas associated with the universal operators that are adjoints of analytic Toeplitz operators lead us to consider the relationship between their invariant subspaces and the invariant subspaces of T_z^*, that is, proper closed invariant subspaces for the backward shift acting on H^2. If L and M are both closed subspaces of H^2, we say the subspace M has non-trivial intersection with L if $(0) \neq L \cap M \neq L$.

One of the most interesting consequences that follows from studying the class of adjoints of analytic Toeplitz operators that are universal in the sense of Rota is the following surprising result, a restatement of Corollary 27.

Theorem. *If every closed, infinite dimensional, invariant subspace for the adjoint of an analytic Toeplitz operator on the Hardy space H^2 that is universal in the sense of Rota has a non-trivial intersection with some invariant subspace of T_z^*, then every bounded linear operator on a separable Hilbert space of dimension two or more has a non-trivial closed invariant subspace.*

We observe that cyclic and non-cyclic vectors for the backward shift in the Hardy space were characterized by R.G. Douglas, H.S. Shapiro and A.L. Shields in a classic paper [11] from 1970 and other results for non-cyclic vectors may be found in work of Ahern and Clark [1] and Herrero and Sherman [15]. In addition to that, as Prof. N. Nikolski has kindly pointed out to us, there exist infinite dimensional closed subspaces consisting only of cyclic vectors for the backward shift. Of course, it is not known if any of these subspaces are invariant for the adjoint of an analytic Toeplitz operator that is universal.

We will close this work with the discussion of such issues along with some open questions that could lead to progress in the solution of the *Invariant Subspace Problem*, including two that get at the existence of *sharp vectors* (see Section 5) for the universal operators we study.

1.1. Notation and framework

We will reserve the word *subspace* for a linear manifold in a Hilbert space that is norm-closed. If T is a bounded operator on a Hilbert space, a subspace M is called *invariant for the operator* T if x in M implies Tx is also in M and we say M is a *proper invariant subspace* if M is not (0) and not the whole Hilbert space. A subspace M is said to be *hyperinvariant* for T if M is an invariant subspace for every bounded operator that commutes with T.

The major work in this paper is set in the Hardy Hilbert space, $H^2(\mathbb{D})$ (also written H^2). Of course, because any two separable, infinite dimensional complex Hilbert spaces are isometrically isomorphic, our choice of H^2 is not limiting in any way. There are two standard definitions for $H^2(\mathbb{D})$: the power series definition is

$$H^2(\mathbb{D}) = \{h \text{ analytic in } \mathbb{D} : h(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \|h\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

If we regard the series for h as a Fourier series $\sum_{n=0}^{\infty} a_n e^{in\theta}$, then we see how $H^2(\mathbb{D})$ can be regarded as the closed subspace of $L^2(\partial\mathbb{D})$ consisting of those functions whose negative Fourier coefficients are all 0.