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Tail probability of a random sum with a heavy-tailed random number and dependent summands ☆

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ABSTRACT

Let $\{\xi, \xi_k : k \geq 1\}$ be a sequence of widely orthant dependent random variables with common distribution F satisfying $E\xi > 0$. Let τ be a nonnegative integer-valued random variable. In this paper, we discuss the tail probabilities of random sums $S_\tau = \sum_{n=1}^{\tau} \xi_n$ when the random number τ has a heavier tail than the summands, i.e. $P(\xi > x)/P(\tau > x) \to 0$ as $x \to \infty$. Under some additional technical conditions, we prove that if τ has a consistently varying tail, then S_τ has a consistently varying tail and $P(S_\tau > x) \sim P(\tau > x/E\xi)$. On the other hand, the converse problem is also equally interesting. We prove that if S_τ has a consistently varying tail, then τ has a consistently varying tail and that $P(S_\tau > x) \sim P(\tau > x/E\xi)$ still holds. In particular, the random number τ is not necessarily assumed to be independent of the summands $\{\xi_k : k \geq 1\}$ in Theorem 3.1 and Theorem 3.3. Finally, some applications to the asymptotic behavior of the finite-time ruin probabilities in some insurance risk models are given.

1. Introduction

We say that a random variable (r.v.) ξ or its distribution G has a consistently varying tail, denoted by $\xi \in \mathcal{C}$ or $G \in \mathcal{C}$, if its tail $\overline{G}(x) = 1 - G(x) > 0$ for all $x > 0$ and

$$\lim_{y \searrow 1} \liminf_{x\to\infty} \frac{\overline{G}(xy)}{\overline{G}(x)} = 1 \quad \text{or equivalently} \quad \lim_{y \nearrow 1} \limsup_{x\to\infty} \frac{\overline{G}(xy)}{\overline{G}(x)} = 1.$$

Clearly, if $G \in \mathcal{C}$, then G has a heavy tail, i.e. $\int_0^\infty e^{tx} G(dx) = \infty$ for all $t > 0$. It is well known that the class $\mathcal{C}$ covers the famous classes $\mathcal{R}_{-\alpha}$ of distributions with regularly-varying tails with index $-\alpha$ for some $\alpha > 0$ in the sense that the relation $\overline{G}(x) = x^{-\alpha} L(x)$ holds for $x > 0$, where $L(x)$ is a slowly varying function (i.e. $\lim_{x\to\infty} \frac{L(xy)}{L(x)} = 1$ for any fixed $y > 0$).

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Throughout this paper, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$; and we write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$.

The main goal of the present paper is to investigate the asymptotic behavior of the tail probability of a random sum $S_\tau = \sum_{k=1}^{\tau} \xi_k$, where the summands $\{\xi, \xi_k : k \geq 1\}$ are a sequence of r.v.s with common distribution F, and the random number τ is a nonnegative integer-valued r.v. with distribution F_τ.

Random sums play an important role in many applied probability fields such as financial insurance, risk theory, queueing theory and so on, and have been well studied in the literature. If both F and F_τ have heavy tails, the relation

$$P(S_\tau > x) \sim E\tau \overline{F}(x) + \overline{F}_\tau(x/E\xi)$$

holds under some additional conditions, see [5,16] and references therein. If F belongs to some subclass of heavy tail distributions and $\overline{F_\tau}(x) = o(\overline{F}(x))$, the relation $P(S_\tau > x) \sim E\tau \overline{F}(x)$ holds under some additional conditions, see [5,8] and references therein. In this paper, we will focus on a case where the random number has a heavier tail than the summands, i.e.

$$\overline{F}(x) = o(\overline{F}_\tau(x)). \tag{1.1}$$

Stam (1973) [11] discussed the case where S_τ had a regularly varying tail. Let $V \in \mathcal{R}_{-\rho}$ for some $\rho > 1$, and suppose that $\tau, \xi_1, \xi_2, \ldots$ are independent and nonnegative. Stam (1973) [11] proved that when

$$\overline{F}(x) = o(\overline{V}(x)), \tag{1.2}$$

then $P(S_\tau > x) \sim c\overline{V}(x)$ for some constant $c > 0$ if and only if $\overline{F}_\tau(x) \sim b\overline{V}(x)$ for some constant $b > 0$ and $c = b(EX)^\rho$. Faÿ et al. (2006) [7] considered the case $0 < \rho \leq 1$. They proved that the above conclusion still held in the case of $0 < \rho < 1$ and $EX < \infty$. But in the case of $\rho = 1$ and $E\tau = \infty$, some stronger conditions than (1.2) are needed. Aleškevičienė et al. (2008) [1] and Robert and Segers (2008) [10] considered the case where τ had a consistently varying tail distribution. Suppose that $\tau, \xi_1, \xi_2, \ldots$ are independent and nonnegative, [1] and [10] proved that if $E\tau < \infty$ and $\overline{F}(x) = o(\overline{F}_\tau(x))$, then $\tau \in \mathcal{C}$ implied that

$$P(S_\tau > x) \sim \overline{F}_\tau(x/E\xi). \tag{1.3}$$

Under some additional technical conditions, [10] proved that the above implication relationship remained true when $E\tau = \infty$. Yang et al. (2009) [15], Zhang et al. (2011) [16] and Cheng and Li (2013) [4] extended the above results to the case where the summands $\{\xi_k : k \geq 1\}$ was a sequence of r.v.s supported on $(-\infty, \infty)$ with some dependence structure.

In the present paper we will consider the case where the summands $\{\xi_k : k \geq 1\}$ are a sequence of widely orthant dependent (see Definition 2.1 below) r.v.s. Under some additional technical conditions, Theorem 3.1 and Theorem 3.2 show that $\tau \in \mathcal{C}$ implies (1.3) and $S_\tau \in \mathcal{C}$. On the other hand, the converse problem is also equally interesting. Theorem 3.3 and Theorem 3.4 show that $S_\tau \in \mathcal{C}$ also implies (1.3), and hence, $\tau \in \mathcal{C}$. In particular, the random number τ is not necessarily assumed to be independent of the summands $\{\xi_k : k \geq 1\}$ in Theorems 3.1 and 3.3. Finally, some applications to the asymptotic behavior of the finite-time ruin probability in some insurance risk models are given.

We will introduce some definitions of dependence structures and a few classes of distributions in Section 2; In Section 3, we give the main results of this paper; In Section 4, we give some applications. The proofs of Theorems 3.1–3.4 are given in Section 5.