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Minimization problem related to a Lyapunov inequality

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ABSTRACT

In this study, we consider the following minimization problem on a bounded smooth domain Ω in $\mathbb{R}^N$:

$$S' := \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \;\Big|\; u \in H^1(\Omega) \setminus \{0\}, \int\limits_\Omega |u|^{2^*-2}u = 0 \right\}.$$

This minimization problem plays a crucial role in the study of L^p-Lyapunov type inequalities ($1 \le p \le \infty$) for linear partial differential equations with Neumann boundary conditions. In this study, we prove the existence of minimizers of S'. As a consequence, we prove the L^p-Lyapunov type inequality in the critical case, which was left open in [4].

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 3$, with a smooth boundary. Let $2^* := 2N/(N-2)$ denote the critical Sobolev exponent. In this study, we consider a minimization problem with a sign-changing condition:

$$S' := \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \;\Big|\; u \in H^1(\Omega) \setminus \{0\}, \int\limits_\Omega |u|^{2^*-2}u = 0 \right\},$$

where $\|u\|_s = (\int_\Omega |u|^s)^{1/s}$ is the usual $L^s(\Omega)$-norm. For $1 \le q < 2^*$, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact. In addition, it is well known that the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is non-compact, and thus the existence of extremals of S' is nontrivial. A minimization problem involving the critical Sobolev exponent

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under a sign-changing condition was studied by Girão and Weth [7], where they also studied the minimization problem:

$$\Lambda_{2^*} := \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \;\middle|\; u \in H^1(\Omega) \setminus \{0\}, \int_\Omega u = 0 \right\}.$$

They proved that an extremal of Λ_{2^*} exists for $N \geq 3$. Furthermore, it was proved that the minimizers belong to $C^{3,\alpha}(\overline{\Omega})$ and if Ω is an open unit ball, then the extremal functions are foliated Schwarz symmetric. In contrast to the problem Λ_{2^*}, the set of admissible functions for S' is not a linear space, which causes some technical difficulties.

The existence of minimizers of S' is related to an L^p-Lyapunov type inequality [4], as follows. Assume that $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ is smooth bounded domain. Consider the linear problem

$$\begin{cases} -\Delta u(x) = a(x)u(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where it is assumed that the function $a : \Omega \to \mathbb{R}$ belongs to

$$\Lambda := \left\{ a \in L^{\frac{N}{2}}(\Omega) \setminus \{0\} \;\middle|\; \int_\Omega a(x)dx \geq 0 \text{ and (1) has nontrivial solutions} \right\}$$

if $N \geq 3$, and

$$\Lambda := \left\{ a \;\middle|\; a \in L^r(\Omega) \setminus \{0\} \text{ for some } r \in (1, +\infty], \int_\Omega a(x)dx \geq 0, \right.$$
$$\left. \text{and (1) has nontrivial solutions} \right\} \quad \text{if } N = 2.$$

For $1 \leq p \leq \infty$, define the value β_p as

$$\beta_p = \inf\left\{ \|a^+\|_{L^p(\Omega)} \;\middle|\; a \in \Lambda \cap L^p(\Omega) \right\}.$$

The eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega \end{cases}$$

belong to Λ, so Λ is not empty and thus Λ is well defined. By the term L^p*-Lyapunov inequality, we refer to the problem of investigating the attainability of* β_p*, as well as determining the value of* β_p. For this problem, Cañada, Montero, and Villegas [4] proved the following result.

Theorem 1.1. *(See [4].)*

(i) *If $N = 2$ and $p = 1$, or $N \geq 3$ and $1 \leq p < \frac{N}{2}$, then $\beta_p = 0$ and β_p is not attained.*
(ii) *If $\frac{N}{2} < p \leq \infty$, then β_p is attained.*
(iii) *If $N \geq 3$ and $p = \frac{N}{2}$, then $\beta_{\frac{N}{2}} > 0$.*

For case (iii) (we refer to this as the critical case), the attainability of $\beta_{\frac{N}{2}}$ has not been studied previously. Thus, in this study, we prove the attainability of $\beta_{\frac{N}{2}}$. The main result of this study is as follows.