J. Math. Anal. Appl. 432 (2015) 531–549

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lie symmetries of birational maps preserving genus 0 fibrations

Mireia Llorens [a,*], Víctor Mañosa [b]

[a] Dept. de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
[b] Dept. de Matemàtica Aplicada III, Control, Dynamics and Applications Group (CoDALab), Universitat Politècnica de Catalunya, Colom 1, 08222 Terrassa, Spain

ARTICLE INFO

Article history:
Received 12 February 2015
Available online 3 July 2015
Submitted by M.J. Schlosser

Keywords:
Integrable maps
Lie symmetries
Periodic orbits
Rational parameterizations

ABSTRACT

We prove that any planar birational integrable map, which preserves a fibration given by genus 0 curves has a Lie symmetry and some associated invariant measures. The obtained results allow to study in a systematic way the global dynamics of these maps. Using this approach, the dynamics of several maps is described. In particular we are able to give, for particular examples, the explicit expression of the rotation number function, and the set of periods of the considered maps.

1. Introduction

A planar *rational* map $F : \mathcal{U} \to \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{K}^2$ is an open set and where $\mathbb{K}$ can either be $\mathbb{R}$ or $\mathbb{C}$, is called *birational* if it has a rational inverse F^{-1} defined in $\mathcal{U}$. Such a map is *integrable* if there exists a non-constant function $V : \mathcal{U} \to \mathbb{K}$ such that $V(F(x, y)) = V(x, y)$, which is called a *first integral* of F. If a map F possesses a first integral V, then the level sets of V are invariant under F. We say that a map F preserves a fibration of curves $\{C_h\}$ if each curve C_h is invariant under the iterates of F.

In this paper, we consider integrable birational maps that have *rational first integrals*

$$V(x, y) = \frac{V_1(x, y)}{V_2(x, y)},$$

so that they preserve the fibration given by the algebraic curves

$$V_1(x, y) - h\, V_2(x, y) = 0, \text{ for } h \in \mathrm{Im}(V).$$

The dynamics of planar birational maps with rational first integrals can be classified in terms of the *genus* of the curves in the fibration associated to the first integral. In summary, it is known that any

* Corresponding author.
E-mail addresses: mllorens@mat.uab.cat (M. Llorens), victor.manosa@upc.edu (V. Mañosa).

http://dx.doi.org/10.1016/j.jmaa.2015.06.069

birational map F preserving a fibration of *nonsingular* curves of generic genus greater or equal than 2 is globally periodic. This is a consequence of Hurwitz automorphisms theorem and Montgomery periodic homeomorphisms theorem. If the genus of the preserved fibration is generically 1, then either F or F^2 is conjugate to a linear action. The reader is referred to [7,8,11] and references therein for more details.

We present a systematic way to study the case where $\{V = h\}_{h\in \mathrm{Im}(V)}$ is a *genus* 0 *fibration*, that is, when each curve $\{V = h\}$ have genus 0. As it is shown in the next section, by using rational parameterizations of the curves of the invariant fibration one obtains that on each curve any birational map is conjugate to a Möbius transformation. Using this fact, we will prove that any of these maps possesses a Lie symmetry with an associated invariant measure. The proofs are constructive, so we are able to give the explicit expression of the symmetry and the density of the measure for particular examples. These results permit us to give a global analysis of the dynamics of the maps under consideration. In particular, this approach allows to obtain the explicit expression of the rotation number function associated to the maps defined on an open set foliated by closed invariant curves. The explicitness of the rotation number function is a special feature of these kind of maps, on the contrary of what happens when the maps preserve genus 1 fibrations, and it facilitates the full characterization of the set of periods of the maps.

The paper is structured in two sections. In Section 2 we present the main results of the paper: the ones that ensures the existence of Lie symmetries and invariant measures, Theorem 2 and Corollary 5 respectively, and the one that characterizes the existence of conjugations of birational maps with conjugated associated Möbius maps, Proposition 6. In Section 3, we apply our approach to study the global dynamics of some particular maps and recurrences.

2. Main results

2.1. Dynamics via associated Möbius transformations

From now on we assume that F is a birational map with a rational first integral $V = V_1/V_2$, where both F and V are defined in an open set of the domain of definition of the dynamical system, the *good set* $\mathcal{G}(F) \subset \mathbb{K}^2$. We also assume that there exists a set $\mathcal{H} \subset \mathrm{Im}(V) \subset \mathbb{K}$ such that the set $\mathcal{U} := \{(x, y) \in \mathbb{K}^2\colon V(x, y) \in \mathcal{H}\} \cap \mathcal{G}(F)$ is a non empty open set of $\mathbb{K}^2$, and each curve $C_h := \{V_1(x, y) - hV_2(x, y) = 0\}$ is irreducible in $\mathbb{C}$ and has genus 0.

A characteristics of genus 0 curves is that they are rationally parameterizable. Recall that a *rational* curve C in $\mathbb{C}^2$, is an algebraic one such that there exist rational functions $P_1(t), P_2(t) \in \mathbb{C}(t)$ such that for almost all of the values $t \in \mathbb{K}$ we have $(P_1(t), P_2(t)) \in C$; and reciprocally, for almost all point $(x, y) \in C$, there exists $t \in \mathbb{K}$ such that $(x, y) = (P_1(t), P_2(t))$. The map $P(t) = (P_1(t), P_2(t))$ is called a *rational parametrization* of C.

The relationship between rational curves and genus 0 ones is given by the Cayley–Riemann Theorem which states that an algebraic curve in $\mathbb{C}^2$ is rational if and only if has genus 0, see [14, Thm. 4.63] for instance.

Of course a rational curve has not a unique rational parametrization, however one can always obtain a *proper* one, that is, a parametrization P such that P^{-1} is also rational. In other words, one can always find a birational parametrization of a genus 0 curve C, which is unique modulus Möbius transformations, [14, Lemma 4.13].

Observe that given a birational map F preserving an algebraic curve C defined in $\mathbb{K}^2$, by using homogeneous coordinates, one can always extend them to a polynomial map $\widetilde{F}$ and an algebraic curve $\widetilde{C}$ in $\mathbb{K}P^2$. Our main tool, is the following known result: