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On dynamical justification of quantum scattering cross section

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A B S T R A C T

We suggest a dynamical justification of quantum differential cross section in the context of long-time transition to stationary regime for the Schrödinger equation. The problem has been stated by Reed and Simon. Our approach is based on spherical incident waves produced by a harmonic source and the long-range asymptotics for the corresponding spherical limiting amplitudes. The main results are as follows: i) the convergence of spherical limiting amplitudes to the limit as the source goes away to infinity, and ii) the proof of the coincidence of the corresponding limit scattering cross section with the universally recognized formula. The main technical ingredients are the Agmon–Jensen–Kato's analytical theory of the Green function, Ikebe's uniqueness theorem for the Lippmann–Schwinger equation, and some refinement of classical long-range asymptotics for the Coulomb potentials.

1. Introduction

The differential cross section is the main observable in quantum scattering experiments. This concept was first introduced to describe the Rayleigh scattering of sunlight and the Rutherford alpha-particle scattering as the quotient

$$\sigma(\theta) = j_a^{\rm sc}(\theta)/|j^{\rm in}|. \tag{1.1}$$

Here, $j^{\rm in}$ is the incident stationary flux, and $j_a^{\rm sc}(\theta)$ is the angular density of the scattered stationary flux $j^{\rm sc}(x)$ in the direction $\theta \in \mathbb{R}^3$, $|\theta| = 1$ (see Fig. 1):

$$j_a^{\rm sc}(\theta) = \lim_{R\to\infty} R^2 j^{\rm sc}(R\theta)\theta. \tag{1.2}$$

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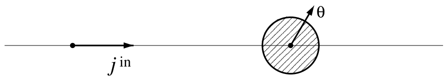

Fig. 1. Incident flux and scatterer.

In both scattering processes studied by Rayleigh and Rutherford the concept of differential cross section is well-established in the framework of the corresponding dynamical equations: the Maxwell equations in the case of Rayleigh scattering and the Newton equations in the case of Rutherford scattering.

On the other hand, a satisfactory dynamical justification of quantum scattering cross section is still missing in the framework of the Schrödinger equation

$$i\dot{\psi}(x,t) = H\psi(x,t) := -\frac{1}{2}\Delta\psi(x,t) + V(x)\psi(x,t), \qquad x \in \mathbb{R}^3. \tag{1.3}$$

The problem has been stated and discussed by Reed and Simon in [19, pp. 355–357]. We suggest the solution for the first time, as far as we are aware. The corresponding charge and flux densities are defined as

$$\rho(x,t) = |\psi(x,t)|^2, \qquad j(x,t) = \mathrm{Im}[\overline{\psi(x,t)}\nabla\psi(x,t)]. \tag{1.4}$$

These densities satisfy the charge continuity equation

$$\dot{\rho}(x,t) + \operatorname{div} j(x,t) = 0, \qquad (x,t) \in \mathbb{R}^4. \tag{1.5}$$

Let us denote by $k \in \mathbb{R}^3 \setminus 0$ the 'wave vector' of the incident plane wave

$$\psi^{\mathrm{in}}(x,t) = e^{i(kx - E_k t)}, \qquad E_k := \frac{1}{2}k^2. \tag{1.6}$$

Our main goal is a dynamical justification of the formula for the differential cross section

$$\sigma(k,\theta) = 16\pi^4 |T(|k|\theta, k)|^2, \qquad \theta \neq \pm n := \pm k/|k|, \tag{1.7}$$

which is universally recognized in physical and mathematical literature (see, for example, [11,17,19,22,25]).

Let the brackets $(\cdot,\cdot)$ denote the Hermitian scalar product in the complex Hilbert space $\mathcal{L}^2 := L^2(\mathbb{R}^3)$, as well as its extension to the duality between the weighted Agmon–Sobolev spaces, see (2.2) and (7.11). The *T-matrix* is given by

$$T(k',k) := \frac{1}{(2\pi)^3}(T(E_k + i0)e^{ikx}, e^{ik'x}), \qquad k', k \in \mathbb{R}^3, \tag{1.8}$$

which is the integral kernel of the operator $T(E_k + i0) := V - VR(E_k + i0)V$ (see Section 25 of [15]) in the Fourier transform

$$\hat{\psi}(k) = \int e^{-ikx}\psi(x)dx, \qquad \psi \in C_0^\infty(\mathbb{R}^3). \tag{1.9}$$

Here, $R(E) := (H - E)^{-1}$ is the resolvent of the Schrödinger operator H.

It is well known that the integral kernel $S(k',k)$ of the scattering operator S in the Fourier transform reads as

$$S(k',k) = \delta(k' - k) - i\pi\delta(E_{k'} - E_k)T(k',k), \quad k', k \in \mathbb{R}^3 \tag{1.10}$$