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The Zakharov–Kuznetsov equation in weighted Sobolev spaces ☆

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A B S T R A C T

In this work we consider the initial value problem (IVP) associated to the two dimensional Zakharov–Kuznetsov equation

$$\left.\begin{aligned} u_t + \partial_x^3 u + \partial_x \partial_y^2 u + u\partial_x u = 0, \qquad & (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y). \qquad & \end{aligned}\right\}$$

We study the well-posedness of the IVP in the weighted Sobolev spaces

$$H^s(\mathbb{R}^2) \cap L^2((1+x^2+y^2)^r dxdy),$$

with $s, r \in \mathbb{R}$.

1. Introduction

In this article we consider the initial value problem (IVP) associated to the two dimensional Zakharov–Kuznetsov (ZK) equation,

$$\left.\begin{aligned} u_t + \partial_x^3 u + \partial_x \partial_y^2 u + u\partial_x u &= 0, \qquad (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ u(x,y,0) &= u_0(x,y). \end{aligned}\right\} \tag{1.1}$$

This equation is a bidimensional generalization of the Korteweg–de Vries (KdV) equation and in three spatial dimensions was derived by Zakharov and Kuznetsov in [31] to describe unidirectional wave propagation in a magnetized plasma. A rigorous justification of the ZK equation from the Euler–Poisson system for uniformly magnetized plasma was done by Lannes, Linares and Saut in Chapter 10 of [17].

Lately, different aspects of the ZK equation and its generalizations have been extensively studied.

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With respect to the local and global well posedness (LWP and GWP) of the IVP (1.1) in the context of classical Sobolev spaces, Faminskii in [4], established GWP in $H^s(\mathbb{R}^2)$, for $s \geq 1$, integer. For that, Faminskii followed the arguments developed by Kenig, Ponce and Vega for the Korteweg–de Vries equation in [16], which use the local smoothing effect, a maximal function estimate and a Strichartz type inequality, for the group associated to the linear part of the equation, to obtain LWP by the contraction mapping principle. Then the global result is a consequence of the conservation of energy. In [18], Linares and Pastor refined Faminskii's method and obtained LWP for initial data in Sobolev spaces $H^s(\mathbb{R}^2)$, for $s > 3/4$. Recently, symmetrizing the ZK equation and using the Fourier restriction norm method (Bourgain's spaces, see [2]), Grünrock and Herr in [10] improved the previous results, establishing LWP of the IVP (1.1) in $H^s(\mathbb{R}^2)$ for $s > 1/2$. Independently, without symmetrization, the same result was obtained by Pilod and Molinet in [21]. For that they used new bilinear Strichartz estimates deduced directly from the original equation. In this manner their method also works for the case of periodic solutions in the space variable y.

LWP and GWP of the IVP (1.1) for the ZK equation and its generalizations also have been considered in the articles [1,5,19,20,25,26] and references therein.

In [14], Kato studied the IVP for the generalized KdV equation in several spaces, besides the classical Sobolev spaces. Among them, Kato considered weighted Sobolev spaces.

In this work we will be concerned with the well-posedness of the IVP (1.1) in weighted Sobolev spaces. This type of spaces arises in a natural manner when we are interested in determining if the Schwartz space is preserved by the flow of the evolution equation in (1.1).

Some relevant nonlinear evolution equations as the KdV equation, the non-linear Schrödinger equation and the Benjamin–Ono equation have also been studied in the context of weighted Sobolev spaces (see [6,7,11–13,22–24] and references therein).

We will study real valued solutions of the IVP (1.1) in the weighted Sobolev spaces

$$Z_{s,r} := H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^r dxdy),$$

with $s, r \in \mathbb{R}$.

The relation between the indices s and r for the solutions of the IVP (1.1) can be found, after the following considerations, contained in the work of Kato: suppose we have a solution $u \in C([0, \infty); H^s(\mathbb{R}^2))$ to the IVP (1.1) for some $s > 1$. We want to estimate (pu, u), where $p(x, y) := (1 + x^2 + y^2)^r$ and $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^2)$. Proceeding formally we multiply the ZK equation by up, integrate over $(x, y) \in \mathbb{R}^2$ and apply integration by parts to obtain:

$$\frac{d}{dt}(pu, u) = -3(p_x \partial_x u, \partial_x u) - (p_x \partial_y u, \partial_y u) - 2(p_y \partial_y u, \partial_x u) + ((p_{xxx} + p_{xyy})u, u) + \frac{2}{3}(p_x u^3, 1).$$

To see that (pu, u) is finite and bounded in t, we must bound the right hand side in the last equation in terms of (pu, u) and $\|u\|^2_{H^s}$. The most significant terms to control in the right hand side in the equation are the three first ones. They may be controlled in the same way. Let us indicate how to bound the first term. Using the Interpolation Lemma 2.5 (see Section 2), for $\theta \in [0, 1]$ and $u \in Z_{s,r}$ we have

$$\|(1 + x^2 + y^2)^{(1-\theta)r/2} u\|_{H^{\theta s}} \leq C\|(1 + x^2 + y^2)^{r/2} u\|^{(1-\theta)}_{L^2} \|u\|^{\theta}_{H^s}.$$

The term $3(p_x \partial_x u, \partial_x u)$ can be controlled when $\theta s = 1$ if

$$|p_x| \leq (1 + x^2 + y^2)^{(1-\theta)r}. \tag{1.2}$$

Taking into account that $|p_x| \leq (1 + x^2 + y^2)^{r-1/2}$, in order to have (1.2) it is enough to require that $r - 1/2 = (1 - \theta)r$. This condition, together with $\theta s = 1$, leads to $r = s/2$.