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Reversible nilpotent centers with cubic homogeneous nonlinearities

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We provide 13 non-topological equivalent classes of global phase portraits in the Poincaré disk of reversible cubic homogeneous systems with a nilpotent center at origin, which complete the classification of the phase portraits of the nilpotent centers with cubic homogeneous nonlinearities.

1. Introduction and statement of the results

One of the main problems in the qualitative theory of planar polynomial differential systems, beside determining their limit cycles and their number, is the center-focus problem, i.e. the problem of distinguishing between a center or a focus. The beginning of this problem goes back to Poincaré, who defined a *center* as a singular point with a neighborhood filled with periodic orbits except the singular point.

It is known that if the polynomial differential system has a center at the origin, then there exists a change of variables and a time rescaling (if necessary) which transforms the original system into one of the following

$$\dot{x} = -y + P(x,y), \quad \dot{y} = x + Q(x,y); \tag{1}$$

$$\dot{x} = y + P(x,y), \quad \dot{y} = Q(x,y); \tag{2}$$

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y); \tag{3}$$

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where $P(x,y)$ and $Q(x,y)$ are polynomials without constant and linear terms. The center of the form (1) is called *linear type center*, of the form (2) *nilpotent center*, and of the form (3) *degenerate center*.

The complete classification of centers of the form (1) for quadratic real polynomial differential systems has been done mostly by Dulac [8], Kapteyn [16,17] and Bautin [2]. The phase portraits of these systems were done by Vulpe [21] and Schlomiuk [20]. We know some classifications of centers for some families of cubic differential systems and of differential systems of higher degree, see [14,19] and the references therein. The normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian linear type centers of linear plus cubic homogeneous planar polynomial vector fields have been given in [6].

In this paper we focus our attention on nilpotent centers. An algorithm that characterizes nilpotent centers and some other classes of degenerate centers has been given in [10,11], see also [15]. It is known that quadratic polynomial differential systems have no nilpotent centers, see for instance [4]. There are some works where the analytic integrability of nilpotent singular points has been studied, see [3–5,12,13].

The objective of this paper is to classify the global phase portraits of the nilpotent centers of the cubic polynomial differential systems of the form

$$\dot{x} = y + Ax^2y + Bxy^2 + Cy^3, \quad \dot{y} = -x + Px^2y + Kxy^2 + Ly^3. \tag{4}$$

Andreev et al. in [1] have obtained the normal forms for system (4) having a nilpotent center, see Theorem 1 of [1]. In fact there are two families of nilpotent centers, the Hamiltonian one, studied in [7], and the reversible family

$$\begin{aligned} \dot{x} &= y + Ax^2y + Cy^3, \\ \dot{y} &= -x^3 + Kxy^2. \end{aligned} \tag{5}$$

Note that this system is invariant, up to a time-reversal, under the symmetries with respect to both axes, and consequently also by the symmetry with respect to the origin. More precisely, the changes of variables $(x,y,t) \to (-x,y,-t)$, $(x,y,t) \to (x,-y,-t)$ and $(x,y) \to (-x,-y)$ leave invariant the system (5). So knowing the phase portrait of the system in one quadrant of the plane, we know completely the phase portrait of the system.

A first integral of system (5) has the form

$$H(x,y) = l_1^{-A-K+\sqrt{D}} l_2^{A+K+\sqrt{D}}, \tag{6}$$

where

$$l_1 = K(A+\sqrt{D}-K)(1+Ax^2) + 2C^2y^2 + C(2+(\sqrt{D}-K)x^2 + A(x^2+2Ky^2))$$

and

$$l_2 = K(-A+\sqrt{D}+K)(1+Ax^2) - 2C^2y^2 + C(-2+(\sqrt{D}+K)x^2 - A(x^2+2Ky^2))$$

are invariant curves of system (5) and $D = (A-K)^2 - 4C$.

For cubic systems with homogeneous nonlinearities with a nilpotent center at the origin until now only the global phase portraits of Hamiltonian centers were done, see [7]. Hence our goal for completing the classification of the nilpotent centers of the cubic polynomial differential systems (4) is to obtain the global phase portraits in the Poincaré disk of systems (5). To do this we will use the Poincaré compactification of polynomial vector fields, see Section 2. Two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing