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Frame sequences and representations for samplable random processes

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ABSTRACT

In this work we characterize random processes which can be linearly determined using a frame sequence of random variables. In particular, these could be the discrete samples of a continuous time process. We study the stable representation of continuous time processes by means of discrete samples or measurements of the original process. Finally, we study how these representations can be applied to reduce the effects of reconstructing a random signal from samples corrupted by additive noise.

1. Introduction

One of the most relevant mathematical problems in communication engineering, and other fields of applications, is the reconstruction of a signal from sampled data. Usually these samples are the values of a function, its derivatives at certain points or other linear operations applied to the function. The fundamental result in sampling theory is the Whittaker–Shannon–Kotelnikov (WSK) theorem [27], which says that if $f \in L^2(\mathbb{R})$ is such that $\hat{f}$ is concentrated in a finite interval $[-B, B]$, then

$$f(t) = \sum_{n\in\mathbb{Z}} \frac{\sin(Bt - \pi n)}{Bt - \pi n} f(\frac{\pi n}{B}), \tag{1.1}$$

where the convergence is uniform and in the L^2-norm. Recent related results and generalizations of this can be found in [4,12].

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In another direction, one generalization of the WSK theorem which, in part, motivates the present work is the Kramer sampling theorem [14,27] for $L^2(I)$ functions:

Theorem 1.1. *Let $k(x,t)$ be a function, defined for all t in a suitable subset D of $\mathbb{R}$ such that, as a function of x, $k(\,.\,,t) \in L^2(I)$ for every $t \in D$, where I is an interval of the real line. Assume that there exists a sequence of distinct real numbers $\{t_n\}_{n\in\mathbb{Z}} \subset D$, such that $\{k(\,.\,,t_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form:*

$$f(t) = \int_I g(x)k(x,t)dx, \tag{1.2}$$

where $g \in L^2(I)$, we have:

$$f(t) = \lim_{N\to\infty} \sum_{|n|\le N} f(t_n)S_n(t),$$

with $S_n(t) = \frac{\langle k(.,t),k(.,t_n)\rangle_{L^2(I)}}{\|k(.,t_n)\|^2_{L^2(I)}}$.

One important fact of the generalization of the deterministic WSK theorem given by Kramer is that it allows to treat the case of non-uniform samples. It is also possible to prove a converse of this result [10]. We study similar conditions for a second order random process or, when possible, in the general setting of an arbitrary (complex) Hilbert space H, considering a general indexed set, or "process": $\{x_t\}_{t\in\mathfrak{T}} \subset H$, indexed by an arbitrary set $\mathfrak{T}$. Kramer's result is strongly related to orthonormal bases, but as noted in [9] and [10], what is really needed is a stability condition, and frame sequences provide an appropriate framework for this. Hilbert space bases provide some natural representations of certain random process [18,6,7]. Moreover these representations are very useful in the analysis of various problems in statistical communication theory, such as detection and estimation [22,3]. Since frame sequences are a natural generalization of the concept of a basis, we shall study conditions for a countable set of samples to be a frame sequence of the Hilbert space spanned by the whole process. The results presented here are a natural generalization of [20], where the case of Riesz bases was studied. As it will be shown, the redundancy present in frame sequences provides a good tool to reduce the effect of additive noise. We shall study different equivalences between several representations [21] for samplable processes. In this context, a samplable process will mean a continuous time, or spatial process, which can be completely linearly determined by a series expansion, using a set of countable samples or measurements of the original process. To motivate this discussion, let us recall the Whittaker–Shannon–Kotelnikov (WSK) sampling theorem for *wide sense stationary processes* (*w.s.s.* processes):

Theorem 1.2. *(See [24].) Let $\mathscr{X} = \{x_t\}_{t\in\mathbb{R}}$ be a w.s.s. random process defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that its spectral measure is concentrated in a finite interval $(-B, B)$, then*

$$x_t = \sum_{n\in\mathbb{Z}} \frac{\sin(Bt - \pi n)}{Bt - \pi n} x_{\frac{\pi n}{B}}, \tag{1.3}$$

where the convergence is in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-norm.

In particular, note that, Eq. (1.3) implies that the process is completely linearly determined by its samples, i.e. $\overline{\text{span}}\{x_k\}_{k\in\mathbb{Z}} = \overline{\text{span}}\{x_t\}_{t\in\mathbb{R}}$. Lloyd [16] gave necessary and sufficient conditions, in terms of the spectral measure, for a w.s.s. process to be completely linearly determined by its samples. This result