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Generalized consistent sampling in abstract Hilbert spaces

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ABSTRACT

We consider generalized consistent sampling and reconstruction processes in an abstract Hilbert space $\mathcal{H}$. We first study the consistent sampling in $\mathcal{H}$ together with its performance analysis. We then study its generalization: partial consistency and quasi-consistency. We give complete characterizations for both of them. We also provide an iterative algorithm to compute the quasi-consistent approximation. An illustrative example is also included.

1. Introduction

A fundamental problem of sampling theory concerns the reconstruction of signals from a discrete set of measurements, namely the samples. Classical results on this problem usually focus on perfect reconstruction of signals under suitable assumptions. The most important example is the Shannon sampling theorem [20] which says that any signal f band-limited in $[-\pi, \pi]$ can be perfectly reconstructed from its uniform samples as $f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(t-n)$, where $\mathrm{sinc}\, t = \frac{\sin \pi t}{\pi t}$ is the cardinal sine function.

More general approach to the problem is to seek an approximation rather than the perfect reconstruction on some restricted class of signals. As a general abstract formulation, consider a Hilbert space $\mathcal{H}$ and a set of measurement vectors $\{v_j\}_{j\in J}$ in $\mathcal{H}$ which span a closed subspace $\mathcal{V}$ (*sampling space*). Given any signal f in $\mathcal{H}$, we take measurements of the form $\{\langle f, v_j\rangle\}_{j\in J}$ and seek an approximation of f as a linear combination of reconstruction vectors $\{w_k\}_{k\in K}$ which span a closed subspace $\mathcal{W}$ (*reconstruction space*). A natural method for such an approximation is to impose the 'consistency', which requires that the approximated signal and the original one yield the same measurements. Consistency was first introduced by Unser and Aldroubi [22] in the setting of shift invariant spaces and later extended significantly by Eldar et al. [7–9,11]. In the beginning, it was assumed [8,9,22] that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ under which we have a unique consistent sampling

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operator given as the oblique projection onto $\mathcal{W}$ along $\mathcal{V}^\perp$. It is proved that for particular shift-invariant sampling and reconstruction spaces $\mathcal{V}$ and $\mathcal{W}$ in $L^2(\mathbb{R})$, this consistent sampling performs well as a sampling approximation process [12,13,23,24]. Later, Hirabayashi and Unser [14] treated the consistent sampling when $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ is finite dimensional and $\mathcal{W} \cap \mathcal{V}^\perp$ may or may not be $\{0\}$. Under such assumptions, they proved that there exist infinitely many consistent sampling operators each of which is an oblique projection. Recently, Arias and Conde [2] studied the problem when $\mathcal{H} \supsetneq \mathcal{W} + \mathcal{V}^\perp$ in which case consistent sampling operators no longer exist. As a generalization of the consistency, they introduced the 'quasi-consistency' which only requires that the measurements of the approximated signal and the original one are as close as possible in l_2 sense. In this work, we first study the consistency in an abstract setting on an arbitrary Hilbert space $\mathcal{H}$ together with its performance analysis. We then study partial consistency and quasi-consistency [2] when the condition $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ is violated. In particular, we show that the quasi-consistency can be also interpreted geometrically in terms of oblique projections. We also provide an iterative method to compute the quasi-consistent approximation.

This paper is organized as follows. In Section 2, we collect definitions and concepts needed throughout the paper. In Section 3, we give complete characterizations of consistent sampling operators along with their performance analysis. Several useful representations of consistent sampling operators are also provided. Section 4 is devoted to two generalizations of the consistency: partial consistency and quasi-consistency, with more focus on the latter. We give complete characterizations of both partial and quasi-consistencies. We also provide an iterative algorithm to obtain the quasi-consistent approximation. Finally we give an illustrative example.

2. Preliminaries

For any countable index set I, let $l_2(I)$ be the set of all complex-valued sequences $\mathbf{c} = \{c(i)\}_{i\in I}$ with $\|\mathbf{c}\|^2 := \sum_{i\in I} |c(i)|^2 < \infty$.

A sequence $\{\phi_n | n \in I\}$ of vectors in a separable Hilbert space $\mathcal{H}$ is

- a frame of $\mathcal{H}$ if there are constants $B \geq A > 0$ such that

$$A\|\phi\|^2_{\mathcal{H}} \leq \sum_{n\in I} |\langle \phi, \phi_n\rangle|^2 \leq B\|\phi\|^2_{\mathcal{H}}, \quad \phi \in \mathcal{H};$$

- a Riesz basis of $\mathcal{H}$ if it is complete in $\mathcal{H}$ and there are constants $B \geq A > 0$ such that

$$A\|\mathbf{c}\|^2 \leq \|\sum_{n\in I} c(n)\phi_n\|^2_{\mathcal{H}} \leq B\|\mathbf{c}\|^2, \quad \mathbf{c} = \{c(n)\}_n \in l_2(I).$$

For any two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $L(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, and $L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$. For any $T \in L(\mathcal{H}, \mathcal{K})$, let $R(T)$ and $N(T)$ be the range and the kernel of T respectively. When $R(T)$ is closed, $T^\dagger$ denotes the Moore–Penrose pseudo-inverse of T [1] and $T^- := \{X \in L(\mathcal{K}, \mathcal{H}) |\ TXT = T\} = \{T^\dagger + Y - T^\dagger TYTT^\dagger | Y \in L(\mathcal{K}, \mathcal{H})\}$ the set of all generalized inverses of T (see Lemma 2.1 below).

Lemma 2.1. *Let $T_1 \in L(\mathcal{V}, \mathcal{K})$ and $T_2 \in L(\mathcal{H}, \mathcal{U})$ have closed ranges, where $\mathcal{H}$, $\mathcal{K}$, $\mathcal{U}$, $\mathcal{V}$ are Hilbert spaces. For any $T \in L(\mathcal{H}, \mathcal{K})$, the equation*

$$T_1 X T_2 = T \tag{1}$$

is solvable for $X \in L(\mathcal{U}, \mathcal{V})$ if and only if $R(T) \subseteq R(T_1)$ and $N(T) \supseteq N(T_2)$. In this case, the general solution X is