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# Gibbs phenomenon for Fourier partial sums on $\mathbb{Z}_p$

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ABSTRACT

It is of interest to know whether the Gibbs phenomenon occurs on a local field. A $p$-adic Heaviside function on the group of $p$-adic integers is defined as an analogy of real variable Heaviside function and it is also shown that there exists the Gibbs phenomenon with an undershoot of at least $1/(p+1)$ as an approximate level. As a consequence of the theorem, we see that the Fourier partial sum for a Heaviside function does not converge at the point of discontinuity.


## 1. Introduction

We begin with the definition of $p$-adic numbers: Let $p$ be a prime number. Every rational number $x$ is represented as

$$x = p^{\gamma}\frac{m}{n}, \tag{1.1}$$

where integers $m$ and $n$ are indivisible by $p$ and $\gamma = \gamma(x)$ is an integer called the valuation of $x\,(\neq 0)$. If the valuation of $x$ is $\gamma$, then the $p$-adic norm $|x|_p$ of $x$ is defined by

$$|0|_p = 0, \quad |x|_p = p^{-\gamma}. \tag{1.2}$$

The field $\mathbb{Q}_p$ is now defined as the completion of the field of rational numbers with respect to the metric induced by the $p$-adic norm and an element of $\mathbb{Q}_p$ is called a $p$-adic number. Here, the $p$-adic norm satisfies the following properties:

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(*i*) $|x|_p \geq 0$, $|x|_p = 0$ if and only if $x = 0$.
(*ii*) $|xy|_p = |x|_p\,|y|_p$.
(*iii*) $|x + y|_p \leq \max(|x|_p, |y|_p)$, moreover, the equality holds when $|x|_p \neq |y|_p$.

For convenience, the canonical representation of $x$ is expressed as

$$x = p^{\gamma} \sum_{j=0}^{\infty} \alpha_j p^j, \tag{1.3}$$

where $\gamma = \gamma(x)$ and $\alpha_j$ are integers such that $0 \leq \alpha_j \leq p - 1$ and the leading coefficient $\alpha_0 \neq 0$. Note that the convergence of (1.3) makes sense only in the $p$-adic norm.

Let $n$ be an integer. For $a \in \mathbb{Q}_p$, a disc and a circle are denoted as $D_{p^n}(a) = \{x : |x - a|_p \leq p^n\}$ and $\partial D_{p^n}(a) = \{x : |x - a|_p = p^n\}$, respectively. Especially, write $\mathbb{Z}_p = D_1(0)$ and it is called the ring of $p$-adic integers. The norm property (*iii*) implies that if $b \in D_{p^n}(a)$, then $D_{p^n}(a) = D_{p^n}(b)$ [44, pp. 8–9].

We employ the following $p$-adic order to define an analogy of real-variable functions [44, p. 14]: For $x = p^{\gamma(x)} \sum_{k=0}^{\infty} x_k p^k$ and $y = p^{\gamma(y)} \sum_{k=0}^{\infty} y_k p^k$, write $x <_p y$ (or $y >_p x$) if either $|x|_p < |y|_p$ or $x_k < y_k$ for the smallest nonnegative integer $k$ with $x_k \neq y_k$ whenever $|x|_p = |y|_p$.

For $a \in \mathbb{Q}_p$, it is said that $f : \mathbb{Q}_p \to \mathbb{R}$ has a point of jump-discontinuity if there exists $a \in \mathbb{Q}_p$ such that $f$ is continuous except at $a$ and

$$\lim_{x \nearrow a} f(x) \;\text{ and }\; \lim_{x \searrow a} f(x) \;\text{ exist,}$$

where $x \nearrow a$ ($x \searrow a$, resp.) means that $x$ increases (decreases, resp.) to $a$ according to the $p$-adic order.

For $a \neq 0 \in \mathbb{Q}_p$, a $p$-adic Heaviside function $H_a$ (as an analogy of the real variable Heaviside function) is defined by

$$H_a(x) = \begin{cases} 0 & \text{if } x \leq_p a \\ 1 & \text{otherwise,} \end{cases}$$

where the symbol $\leq_p$ means $<_p$ or $=$. In Theorem 3.1 of Section 3, it is shown that $H_a(x)$ has a point of jump-discontinuity at $a$.

Since $\mathbb{Q}_p$ is a locally compact abelian group with respect to addition, there exists normalized Haar measure $dx$, i.e., the measure of $\mathbb{Z}_p$ is 1.

**Main Theorem.** *Let $0 \neq |a|_p < 1$. The $n$-th partial sum $S_n H_a(x)$ of the Fourier series for $H_a$, produces the Gibbs phenomenon at $a$ with an undershoot of at least $1/(p + 1)$ as an approximate level.*

Since about twenty years ago, it has been studied extensively on $p$-adic structures in various models of physics and there have been exciting achievements exploring it. Most of all, there has been significant progress in non-Archimedean modelings of some topics in $p$-adic mathematical physics, for instance, refer to [8,27,44].

The first step toward quantum mechanics with wave functions valued in non-Archimedean fields was done by Vladimirov and Volovich through the view point of mathematical physics and Khrennikov [27] extends their approach in collaboration with Cianci and Albeverio [2,3].

On the other hand, the convergence of classical Fourier series on an interval of the real line has been studied. Especially, for a function having a jump-discontinuity, the Gibbs phenomenon, an overshoot (or an undershoot), occurs in its Fourier series, where the overshoot (or the undershoot) means the positive distance between the function and its Fourier partial sums along some sequence which converges to a point