J. Math. Anal. Appl. 433 (2016) 416–433

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Limiting spectral distribution of Gram matrices associated with functionals of β-mixing processes

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A R T I C L E I N F O

Article history:
Received 23 June 2014
Available online 31 July 2015
Submitted by M. Peligrad

Keywords:
Random matrices
Sample covariance matrices
Stieltjes transform
Absolutely regular sequences
Limiting spectral distribution
Spectral density

A B S T R A C T

We give asymptotic spectral results for Gram matrices of the form $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$ where the entries of $\mathcal{X}_n$ are dependent across both rows and columns. More precisely, they consist of short or long range dependent random variables having moments of second order and that are functionals of an absolutely regular sequence. We also give a concentration inequality of the Stieltjes transform and we prove that, under an arithmetical decay condition on the β-mixing coefficients, it is almost surely concentrated around its expectation. Applications to examples of positive recurrent Markov chains and dynamical systems are also given.

1. Introduction

For a random matrix $\mathcal{X}_n \in \mathbb{R}^{N\times n}$, the study of the asymptotic behavior of the eigenvalues of the $N \times N$ Gram matrix $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$ gained interest as it is employed in many applications in statistics, signal processing, quantum physics, finance, etc. In order to describe the distribution of the eigenvalues, it is convenient to introduce the empirical spectral measure defined by $\mu_{n^{-1}\mathcal{X}_n\mathcal{X}_n^T} = N^{-1}\sum_{k=1}^N \delta_{\lambda_k}$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$. This type of study was actively developed after the pioneering work of Marčenko and Pastur [11], who proved that under the assumption $\lim_{n\to+\infty} N/n = c \in (0, +\infty)$, the empirical spectral distribution of large dimensional Gram matrices with i.i.d. centered entries having finite variance converges almost surely to a non-random distribution. The limiting spectral distribution (LSD) obtained, i.e. the Marčenko–Pastur distribution, is given explicitly in terms of c and depends on the distribution of the entries of $\mathcal{X}_n$ only through their common variance. The original Marčenko–Pastur theorem is stated for random variables having moments of fourth order; for the proof with second moments only, we refer to Yin [18].

Since then, a large amount of study has been done aiming to relax the independence structure between the entries of $\mathcal{X}_n$. For example, Bai and Zhou [2] treated the case where the columns of $\mathcal{X}_n$ are independent

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http://dx.doi.org/10.1016/j.jmaa.2015.07.064
0022-247X/

with their coordinates having a very general dependence structure and moments of fourth order. Recently, Banna and Merlevède [3] extended along another direction the Marčenko–Pastur theorem to a large class of weakly dependent sequences of real random variables having moments of second order. Letting $(X_k)_{k\in\mathbb{Z}}$ be a stationary process of the form $X_k = g(\cdots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \ldots)$, where the ε_k's are i.i.d. real-valued random variables and $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function, they consider the $N \times N$ sample covariance matrix $\mathbf{A}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{X}_k\mathbf{X}_k^T$ with the $\mathbf{X}_k$'s being independent copies of the vector $\mathbf{X} = (X_1, \ldots, X_N)^T$. Assuming that X_0 has just a moment of second order, then provided that $\lim_{n\to\infty} N/n = c \in (0, \infty)$, they prove, under a mild dependence condition, that almost surely, $\mu_{\mathbf{A}_n}$ converges weakly to a non-random probability measure μ whose Stieltjes transform satisfies an integral equation depending on c and on the spectral density of the underlying stationary process $(X_k)_{k\in\mathbb{Z}}$. In a recent paper, Merlevède and Peligrad [12] extend this result to stationary sequences satisfying mild regularity conditions and prove that the empirical spectral measure of a sample covariance matrix generated by independent copies of a stationary regular sequence has a limiting distribution depending only on the spectral density of the sequence.

In the above mentioned model, the random vector $\mathbf{X} = (X_1, \ldots, X_N)^T$ can be viewed as an N-dimensional process repeated independently n times to obtain the $\mathbf{X}_k$'s. However, in practice it is not always possible to observe a high dimensional process several times. In the case where only one observation of length Nn can be recorded, it seems reasonable to partition it into n dependent observations of length N, and to treat them as n dependent observations. Up to our knowledge this was first done by Pfaffel and Schlemm [13] who showed that this approach is valid and leads to the correct asymptotic eigenvalue distribution of the sample covariance matrix if the components of the underlying process are modeled as short memory linear filters of independent random variables. They consider Gram matrices having the same form as in (2.3) and associated with a stationary linear process $(X_k)_{k\in\mathbb{Z}}$ with independent innovations having finite fourth moments and such that the coefficients decay with an arithmetical rate, and they derive its LSD.

In this work, we study the same model of random matrices as in [13] but considering the case where the entries come from a non-causal stationary process $(X_k)_{k\in\mathbb{Z}}$ of the form $X_k = g(\cdots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \ldots)$ where $(\varepsilon_k)_{k\in\mathbb{Z}}$ is an absolutely regular sequence and $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function such that X_k is a proper centered random variable having finite moments of second order. We prove in Theorem 2.1 a concentration inequality for the Stieltjes transform which allows us to prove that, under an arithmetical decay condition on the β-mixing coefficients, the Stieltjes transform is concentrated almost surely around its expectation as n tends to infinity. Having reduced the study to the expectation of the Stieltjes transform, it is enough to show that the latter converges to the Stieltjes transform of a non-random probability measure. This can be achieved by approximating it by the expectation of the Stieltjes transform of a Gaussian matrix having a close covariance structure as shown in Theorem 2.2. Finally, provided that the spectral density of $(X_k)_k$ exists, we prove in Theorem 2.3 that almost surely, $\mu_{\mathbf{B}_n}$ converges weakly to the same non-random limiting probability measure μ obtained in the cases mentioned before.

We recall now that the absolutely regular (β-mixing) coefficient between two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\beta(\mathcal{A}, \mathcal{B}) = \frac{1}{2}\sup\Big\{\sum_{i\in I}\sum_{j\in J}\big|\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)\big|\Big\},$$

where the supremum is taken over all finite partitions $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ that are respectively $\mathcal{A}$ and $\mathcal{B}$ measurable (see Rozanov and Volkonskii [16]). The coefficients $(\beta_n)_{n\geqslant 0}$ of a sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ are defined by

$$\beta_0 = 1 \quad \text{and} \quad \beta_n = \sup_{k\in\mathbb{Z}} \beta\big(\sigma(\varepsilon_\ell,\ \ell \leqslant k),\ (\varepsilon_{\ell+n},\ \ell \geqslant k)\big) \quad \text{for} \quad n \geqslant 1. \tag{1.1}$$

Moreover, $(\varepsilon_i)_{i\in\mathbb{Z}}$ is said to be absolutely regular or β-mixing if $\beta_n \to 0$ as $n \to \infty$.