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Nontrivial solutions for Kirchhoff-type problems with a parameter [☆]

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ABSTRACT

The paper deals with the following Kirchhoff-type problems:

$$\begin{cases} -(a+b\int_{\mathrm{R}^N}|\nabla u|^2dx)\triangle u+\lambda V(x)u=f(x,u) \text{ in } \mathrm{R}^N, \\ u\in H^1(\mathrm{R}^N), \end{cases}$$

where constants $a, b > 0$, $\lambda \geq 1$ is a parameter, and $f(x,u)$ is either sublinear or superlinear at infinity. Under relaxed assumptions on $V(x)$, we establish the existence and multiplicity results without the compactness of embedding of the working space.

1. Introduction

Consider the following Kirchhoff-type problems:

$$\begin{cases} -(a+b\int_{\mathrm{R}^N}|\nabla u|^2dx)\triangle u+\lambda V(x)u=f(x,u) \text{ in } \mathrm{R}^N, \\ u\in H^1(\mathrm{R}^N), \end{cases} \tag{P_λ}$$

where $N \geq 3$, constants $a, b > 0$, $\lambda \geq 1$ is a parameter and the potential $V(x)$ satisfies the following hypotheses:

(v_1) $V(x) \in C(\mathrm{R}^N, \mathrm{R})$ satisfies $V(x) \geq 0$ on R^N;
(v_2) there exists $d > 0$ such that the set $\{x \in \mathrm{R}^N : V(x) \leq d\}$ has finite measure.

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Kirchhoff-type problems are related to the stationary analogue of the equation

$$u_{tt} - (a + b \int_\Omega |\nabla u|^2 dx) \triangle u = f(x, u) \text{ in } \Omega,$$

where u denotes the displacement, $f(x, u)$ the external force, and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type arise in the study of string or membrane vibration and were first proposed by Kirchhoff in 1883 (see [8]) to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω, which provokes some mathematical difficulties. Similar nonlocal problems also model several physical and biological systems, where u describes a process which depends on the average of itself, for example, the population density; see [5,6], and the references therein.

Recently, under the following conditions on the potential $V(x)$:

(v_1') $V(x) \in C(\mathrm{R}^N, \mathrm{R})$ satisfies $\inf\limits_{x \in \mathrm{R}^N} V(x) \geq b_1 > 0$, where $b_1 > 0$ is a constant.

(v_2') for any $M > 0$, $\mathrm{meas}\{x \in \mathrm{R}^N : V(x) \leq M\} < \infty$, where meas(.) denotes the Lebesgue measure in R^N, the authors proved the existence and multiplicity of nontrivial solutions of (P_1), see, for example, [3,16]. We emphasize that hypotheses (v_1') and (v_2') were used in Bartsch and Wang [1] to guarantee the compact embedding of the working space (see [18, Lemma 3.4]). It is obvious that (v_2) is more weaker than (v_2'). If replacing (v_2') by the condition (v_2), the compactness of the embedding fails, and this situation becomes more delicate. More recently, some authors of [9,12,15] deal with this cases. They showed the existence and multiplicity results of (P_λ) when (v_1)–(v_2) holds and the nonlinearity f is critical, subcritical or superlinear at infinity. Particularly, Sun and Wu [15] obtained the existence of ground state solutions.

Motivated by the works just described, in this paper, under more general conditions on $V(x)$, we consider the existence and multiplicity of nontrivial solutions of the problem (P_λ). In order to state our main results, we first make the following assumptions on f and its primitive function $F(x, u) := \int_0^u f(x, s)ds$.

(f_1) There exist constants $1 < \gamma_1 < \gamma_2 < \ldots < \gamma_m < 2$ and functions $h_i(x) \in L^{2/2-\gamma_i}(\mathrm{R}^N, \mathrm{R}^+)$ ($i = 1, 2, \ldots, m$) such that

$$|f(x, u)| \leq \sum_{i=1}^{m} h_i(x)|u|^{\gamma_i - 1}.$$

(f_2) There exists a nonzero measure open set $J \subset \mathrm{R}^N$ and three constants δ, $\nu > 0$ and $1 < \gamma_0 < 2$ such that

$$F(x, u) \geq \nu |u|^{\gamma_0}, \ \forall (x, u) \in J \times [-\delta, \delta].$$

(f_3) There exists a nonzero measure open set $J \subset \mathrm{R}^N$ and three constants δ, $\nu > 0$ and $1 < \gamma_0 < 2$ such that

$$uf(x, u) \geq \nu |u|^{\gamma_0}, \ \forall (x, u) \in J \times [-\delta, \delta].$$

(f_4) $f \in C(\mathrm{R}^N \times \mathrm{R}, \mathrm{R})$ and $|f(x, u)| \leq a_1(1 + |u|^{p-1})$ for some $2 < p < 2^*$ and $a_1 > 0$, where $2^* = \frac{2N}{N-2}$.